Lie symmetry method
for partial differential equations
with application in finance

FINAL THESIS

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"Per cui, quella sera,
nell’ansa che il fiume descrive costeggiando la zona degli orti per anziani,
alcuni pensionati avevano fiduciosamente messo a mollo le lenze."
Preface

Partial differential equations (PDE) arise in many areas of scientific research and they have been classically used in describing models spanning from quantum mechanics to biological systems. More recently the theory of PDE have attracted an increasing interest by mathematicians and practitioners working in those areas related to banks and insurances.

When we study a particular differential equation, we have to face with two fundamental questions: "Is there (at least) a solution?" and (just in case) "Is it possible to write it (them) explicitly?" Unfortunately it does not exist a standard procedure to follow to reach these goals. Even if we are able to show that a given differential problem has solutions, we often cannot show them analytically. This makes the derivation of exact solutions for PDE a corner stone in studying differential equations theory. In fact, although numerical analysis has made impressive strides in recent years, still the quest of find analytic solution is the main goal in studying PDE. It follows that a method able to recursively produce solutions of a given PDE may be very useful.

In this thesis we deeply analyse one of the most fruitful technique for finding exact solutions of a given differential equations. Such a method, first developed by the Norwegian mathematician Sophus Lie (1842-1899) in the second half of the XIX century, unifies and extends ad hoc techniques in order to produce exact solutions for PDE by exploiting the fact that many phenomena in nature has internal symmetries. Lie developed a systematic way for finding analytic solution for a wide class of PDEs through continuous transformation groups. These groups are usually called Lie groups. These groups are both a "weird" and startling merger of algebraic group, topological structures and elements of analysis. The core idea is to use internal symmetries admitted by a given PDE in order to reduce the number of independent variables. Thus a solution of the given PDE can be found solving a different differential equations with fewer independent variables.

The Lie work was inspired by Galois’s theory for polynomial equations. In particular Lie tried to create a unified theory of integration for ordinary differential equations similar to the Abelian theory developed to solve algebraic equations. For almost 100 years nobody has further studied the theory. It was in the mid 90’s that Ovsiannikov, in the ex URSS, and Bluman, in the West, have brought back the theory to light. Since then a number of works have appeared. This is mainly due to the high algorithmic complexity characterizing the procedure which gives the internal symmetries of a given PDE and, thus, its solutions. A great improvement in concrete applications of the Lie approach have
been realized by the use of symbolic calculus and fast calculators. Modern computers can be easily programmed to find the desired symmetries, making the theory even more appealing and effective.

In recent years, particularly after the worldwide financial crisis of 1987, the field of mathematical finance has seen a massive development leading it to became one of the most studied field in applied mathematics. Particular attention have been attracted by problems related to the fair pricing procedure for a huge pletora of structured financial instruments such in the case of financial derivatives. Financial derivatives are particular type of contracts whose value depends on an underlying asset. The time behaviour of the price of such a contract is usually modeled by a partial differential equations, eventually driven by stochastic terms. In such a scenario is clearly of particular relevance the existence of a (possibly) unique fair price. Moreover, when such a fair price is shown to exist, is of great importance to explicitly write it as an analytical function of its (eventually stochastic) parameters. This is why many mathematicians have applied the Lie’s theory to obtain efficient ways to price a wide class of financial instruments. We will see further (see Sec. [1] and [2]) how such techniques can be exploited to derive not only analytical solutions but also fundamental solutions and transition density functions associated to particular financial models.

Since the huge realm of applications of the Lie approach, we are sadly not able to treat all possible applications even restricting ourselves to the financial applications. In particular we have chosen to focus our attention on linear PDE’s problems with one single dependant variable. Particular attention have been given to local symmetries with emphasis on point transformations, i.e. symmetries defined by infinitesimal transformations whose infinitesimals depend on independent variable, dependent variable and its derivatives.

We would like to underline that one of the main reason for the huge success encountered by Lie theory during recent years, relies on the fact that it provides a unified approach to treat basically any kind of PDE. It follows that in this thesis we are able to merely scratch the surface of a wider theory. In fact symmetries more general than point symmetries, e.g. contact symmetries, Lie-Backlund symmetries, etc., have been developed. Further non-local symmetries i.e. symmetries whose infinitesimals at any point \( x \) depend on the global behaviour of \( u(x) \), are largely used and perhaps they are the area where Lie groups are actually an improvement with respect to standard approaches to PDE’s. In addition system of PDE’s depending on more than just one variable, can be successfully studied and non-linear differential equations are the perfect application of such a method.

**Outline of thesis**

Actually mainly three (slightly) different approaches to the Lie groups theory do exist which differ each one from another by emphasising a different characteristic of the theory. In this work we aim at both unifying different notations coming from such approaches and show how Lie theory of group invariants can be applied for widely used financial models described by stochastic differential equations.

The first approach is due to Ovsiannikov, see Ovsiannikov [Ovs82]. It gives a rigorous mathematical foundation of the theory which is based on both representation theory and transformation groups analysis. During recent years a former student of him, Ibragimov, improved Ovsiannikov
work in a series of works, see e.g. Ibragimov [Ibr09], Ibragimov et al. [Ibr+94; Ibr+95], Ibragimov et al. [Ibr+96], and Ibragimov [Ibr94], related to both the mathematical theory of continuous transformations and some of their application to mathematical physics.

Independently and simultaneously to Ovsiannikov, Bluman studied Lie’s theory in the USA. His original work, see Bluman and Cole [BC74], (together with the one made by Ovsiannikov, see Ovsiannikov [Ovs82]) has been the first that brought back methods associated to Lie groups theory to the attention of the mathematical community. Works by Bluman, see e.g. Bluman and Kumei [BK89], Bluman, Cheviakov, and Anco [BCA10], and Bluman and Anco [BA02], are focused on similarity reduction providing a highly efficient approach to find invariant solutions and mapping of a wide set of differential equations.

This thesis follows anyway a different approach mainly based on a geometric point of view, with a emphasis on differential geometry techniques. This approach can be found in the works of Olver, particularly in his book Olver [Olv93]. A part from the intuitive approach and examples helping understand properly the theory, this book presents one of the best section exploiting the algorithm for finding admitted symmetries. A good presentation of the topic can be found on Olver’s home page http://www.math.umn.edu/~olver/sm.html.

Furthermore it has to be said that, as it was easily understandable, such a theory has an algebraic foundation. The topic is broad and entire books are devoted to it. We do not treat any of such a theory in this thesis since it is beyond our purposes. For the interested reader we refer anyway to Hall [Hal03] and Varadarajan [Var84].

Chapter § 1 is devoted to the theory developed by Lie. At first (almost) all preliminary notions fundamental to the understanding of the method are stated. In section 1.1 we will introduce the fundamental tools such as Lie groups, flows, vector fields, infinitesimal generator and Lie algebras. A complementary part has been introduced in appendix A.1.5. The appendix is not necessary to the understanding of the theory but for an interested reader it better explains the mathematical foundation of the method. Section 1.2 treat the algorithm for finding admitted symmetries. It is the main reason why such a method has seen such a huge development in the past years. The notion of prolongation, Jet-space and total derivative will be introduced. Then the algorithm itself is exploited. Eventually the main results to find any admitted symmetric group of a given PDE are stated. Section 1.3 will focus on the concept of invariant (or self-similar) solutions. It is widely explained how to construct invariant solutions for a PDE. The notions of invariant surface, invariant curve and invariant point will introduced. Section 1.4 treat how to construct a mapping from a given DE to a target DE. In particular such a method it is largely used in finance. For instance the well known Black&Scholes equation can be mapped into the heat equation. Eventually in section 1.5 an example is provided. Everything said will be applied to the Heat equation in order to give an overview on Lie’s theory of symmetry groups. The main references for this chapter are Olver [Olv93], Ibragimov [Ibr09], Bluman and Kumei [BK89], and Ovsiannikov [Ovs82].

In chapter § 2 we will focus on some papers that have been published recently. They use the symmetric group of a certain class of PDE (in particular we will focus on the square root process) in order to retrieve fundamental solutions. The basic idea of the method is to choose a particular parameter $\epsilon$ such that the corresponding Lie group can be treated reduce to a Laplace transform. In
this way the transition density of a given SDE can be derived via inversion of the Laplace transform of a fundamental solution. Section 2.1 is devoted to the statement of some general results that link SDE and parabolic PDE and to some preliminary notions on the method used by Craddock. Section 2.2 will treat instead some particular case of square root process that arise in finance. Furthermore such a section presents some necessary theorem to the derivation of fundamental and analytic solutions as much as transition density functions. Complementary to this chapter is appendix B.1.3 where fundamental solutions are introduced. Again a particular attention is given to the heat equation where its fundamental solution is found. The main references for this chapter are Craddock [Cra09], Craddock and Lennox [CL09; CL07], Craddock and Platen [CP09; CP04; CP03], Craddock and Lennox [CL12], Pascucci [Pas11a], and Friedman [Fri64].

Eventually chapter §3 is devoted to the application of the results stated till then to financial markets. In particular we will focus on three widely used models: the The Cox-Ingersoll-Ross (CIR) model in section 3.1, the constant elasticity of variance (CEV) model in section 3.2 and the stochastic alpha-beta-rho (SABR) model in section 3.3. Pricing of zero coupon bond and options are shown. Since this is a mathematical thesis it is not our intention to extensively treat financial and economical aspects. Basic concept will be introduced to give a mild idea of what it is done but any deeper concepts will be skipped. For a good treatment of financial mathematics we refer to Brigo and Mercurio [BM06], Cvitanić and Zapatero [CZ04], Lamberton and Lapeyre [LL08], Pascucci [Pas11a], and Shreve [Shr04]. For application to pricing we refer to Chen and Lee [CL10], Chuang, Hsu, and Lee [CHL10], Hsu, Lin, and Lee [HLL08], and Sinkala, Leach, and O’Hara [SLO08b; SLO08a]. In appendix C.1 the Longstaff model is treated.
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Introduction

Lie’s method of group invariant has been introduced by Lie in the middle of the XIX century. Lie groups arose in connection with the problem of finding solutions of a given differential equation by quadratures also determining continuous transformation groups. Such approach provides families of solutions, for the assigned differential equation, by a given one via symmetry group. Classical Lie theory consider continuous transformation group which acts on the system’s graph space of dependent and independent variables of a given partial differential equation (PDE). If a differential equation is invariant under a point symmetry, hence it is possible to find similarity solutions which are invariant under subgroups of the whole group. A key point of the described procedure relies on the fact that symmetries of interest can be algorithmically determined.

Briefly a group $G$ having the structure of an analytic manifold $M$ such that the operation $\mu : (x, y) \rightarrow xy^{-1}$ is analytic is called a Lie group. A Lie groups is naturally determined as transformation group and it can be identified with the smooth action of a Lie group on a manifold such that any point is mapped into another point of the same manifold. Moreover local properties of such transformation are sufficient to characterize the transformation globally. In the following we will refer equivalently to Lie group or Lie group of transformations.

Lie’s idea was to solve differential equations by means of internal symmetries admitted by the given equation. A symmetry of a differential equation is a transformation which maps any solution to another solution of the DE. Given a solution of a certain differential equation, knowledge of an admitted symmetry leads to the generation of another solution. In particular given a differential equation $F = 0$ and being $S$ the set of all its solutions, then it can be shown that the collection of all transformations mapping any solution of $F = 0$ into a solution of the same DE, determines a group which is called the group admitted by the differential equation, or the symmetry group of the equation. We will focus in this thesis on continuous transformations.

Previous approach leads to the concept of continuous transformation groups. Multi-parameter continuous transformation groups are composed by one-parameter groups depending on a single continuous parameter. Each one-parameter group is determined by its infinitesimal transformation of the corresponding first-order linear differential operator. In particular we will show (see Sec. 1.1) the analogy between Lie group of transformations and flows of vector fields. One-parameter transformation groups and their generators are connected by means of the so-called Lie equations.
The generators of multi-parameter transformation groups form specific linear spaces known as Lie algebras. Namely, the generators of continuous groups admitted by a given differential are defined by solving an over-determined system of linear differential equations known as the determining equations. The fundamental insight due to Lie was that the symmetry conditions for each of the one-parameter group of transformations composing a multi-parameter group of point transformations turn out to be linear in the infinitesimals of the transformations. Being linear, these infinitesimal determining equations are much easier to analyse and, in a large number of cases, have been solved explicitly to yield the admitted group of point symmetries. The characteristic property of determining equations is that the totality of their solutions span a Lie algebra.

Lie theory is particularly powerful since it allows to reduce the study of complex Lie groups to the study of purely algebraic object, namely their associated Lie algebra by the Lie theorem. We have said how we can linearizing the transformation around the identity $\epsilon = 0$ studying only its infinitesimal generator. Thus an $r$-dimensional Lie group can be studied looking only at its infinitesimal generators $v_1, \ldots, v_r$. The space $V$ generated by the infinitesimal generators $(v_1, \ldots, v_r)$ is the associated Lie algebra i.e. the vector space spanned by the infinitesimal generators and it determines the local structure of the group. Notice that since, roughly speaking, a Lie group is a manifold, it makes perfectly sense to talk about tangent space $V$ around the identity of the group $G$.

If we are dealing with a PDE of the form

$$F(x, u^{(m)}) = 0$$

we would like to find any one-parameter Lie group of transformations leaving invariant the surface determined by the PDE. Lie theory allows us to study the admitted transformations locally, i.e. by considering the infinitesimal generator of the transformation

$$\begin{cases}
\tilde{x} = X_\epsilon(x, u) &= x + \epsilon\xi(x, u) + O(\epsilon^2) \\
\tilde{u} = U_\epsilon(x, u) &= u + \epsilon\phi(x, u) + O(\epsilon^2)
\end{cases} \tag{1}$$

which continuously depends on the parameter $\epsilon$. System (1) is known as the Lie point symmetry group admitted by a PDE.

The set of all such transformations forms a continuous group called the Lie group $G$. According to the theory developed by Lie, the construction of the symmetry group $G$ is equivalent to the determination of an operator, the infinitesimal generator, of the group $G$ having the following form

$$v = \xi(x, u) \partial_x + \phi(x, u) \partial_u.$$

The set of all infinitesimal generators generates the Lie algebra. Such an algebraic structure completely determines the global structure of the whole Lie group.

The group transformations corresponding to the infinitesimal transformations with the infinitesimal
generator are obtained solving the Lie equations

\[
\begin{align*}
\frac{d\tilde{x}}{d\epsilon} &= \xi(x, u), \quad \tilde{x}|_{\epsilon=0} = x, \\
\frac{d\tilde{u}}{d\epsilon} &= \phi(x, u), \quad \tilde{u}|_{\epsilon=0} = u.
\end{align*}
\] (2)

Moreover one can prove that the PDE associated to (2) is left invariant under a certain transformation if the invariance condition

\[
\text{pr}^{(m)} v[F(x, u^{(m)})]|_{F(x, u^{(m)})=0}
\]

where pr\(^{(m)}\)v denotes the prolongation of the vector field, is satisfied.

The invariance condition leads to a set of linear partial differential equations called the determining equations whose solutions give the symmetries admitted by the original PDE. The last step is what makes the method easily applicable in huge set od different fields, spanning from Mathematical Physics, to Finance, etc.
Chapter 1

Lie Group and Differential Equations

1.1 Lie Groups

In this chapter we develop the fundamental theory that lies behind Lie’s theory of continuous transformations group. In particular Lie consider all those transformations that can be represented by the flow of an infinitesimal generator. In our work we will see that these transformations are essentially the flow of a vector field on manifold \( M \). To a given infinitesimal generator can therefore be associated a one-parameter Lie group of points transformations. Moreover from a set of infinitesimal generators arise a multi-parameter Lie group if and only if they form a Lie algebra.

1.1.1 Introduction

We start by recalling two definitions that might look at first sight uncorrelated: the algebraic definition of group and the geometric definition of manifold.

They are both fundamental in the developing of Lie’s theory of continuous transformations.

**Definition 1.1.1 (Group).** A group is a set \( G \) together with a group operation \( (G, \circ) \) such that the following hold:

(i) \( g \circ h \in G, \forall g,h \in G; \)
(ii) \( g \circ (h \circ k) = (g \circ h) \circ K, \forall g,h,k \in G; \)
(iii) \( \exists e \in G : e \circ g = g \circ e = g, \forall g \in G; \)
(iv) \( \forall g \in G \exists g^{-1} \in G : g \circ g^{-1} = e = g^{-1} \circ g. \)

A manifold is an object which, locally, just looks like an open subset of Euclidean space, but whose global topology can be quite different.

**Definition 1.1.2 (Manifold).** An \( m \)-dimensional manifold is a set \( M \) together with a countable collection of subsets \( U_\alpha \subset M \), called coordinate charts, and one-to-one functions \( \chi_\alpha : U_\alpha \to V_\alpha \) onto connected open subsets \( V_\alpha \subset \mathbb{R}^m \), called local coordinate maps, which satisfies the following:
(i) $\bigcup_{\alpha} U_{\alpha} = M$;

(ii) $\chi_{\beta} \circ \chi_{\alpha}^{-1} : \chi_{\alpha}(U_{\alpha} \cap U_{\beta}) \to \chi_{\beta}(U_{\alpha} \cap U_{\beta})$ is a smooth function;

(iii) if $x \in U_{\alpha}$ and $\tilde{x} \in U_{\beta}$ are two distinct points of $M$, then there exist two open subsets $W \subset V_{\alpha}$ and $\tilde{W} \subset V_{\beta}$ with $\chi_{\alpha}(x) \in W$ and $\chi_{\beta}(\tilde{x}) \in \tilde{W}$ such that

$$\chi_{\alpha}^{-1}(W) \cap \chi_{\beta}(\tilde{W}) = \emptyset.$$ 

1.1.2 Basic results on manifolds

The main characteristic of manifolds is that they are coordinates free. In fact besides the basic coordinate charts provided in the definition of a manifold, one can always adjoin many additional coordinate charts $\chi : M \to V \subset \mathbb{R}^m$ subject to the condition that, where defined, the corresponding overlap maps $\chi_{\alpha} \circ \chi_{\beta}^{-1}$ satisfy the same smoothness or analyticity requirements as $M$ itself. The previous concept of change of coordinates is of paramount importance in differential geometry and will be used all over the thesis. For instance, composing a given coordinate map with any local diffeomorphism, meaning a smooth, one-to-one map defined on an open subset of $\mathbb{R}^m$, will give a new set of local coordinates. Often one expands the collection of coordinate charts to include all possible compatible charts, the resulting maximal collection defining an atlas on the manifold $M$. The points in the coordinate chart $W_{\alpha}$ are identified with their local coordinate expressions $x = (x_1, \ldots, x_m) \in V_{\alpha}$. The changes of coordinates provided by the overlap maps are then given by local diffeomorphisms $y = \eta(x)$ defined on the overlap of the two coordinate charts.

A map $F : M \to N$ between smooth manifolds is called smooth if it is smooth in local coordinates. In other words, given local coordinates $x = (x_1, \ldots, x_m)$ on $M$, and $y = (y_1, \ldots, y_n)$ on $N$, the map has the form $y = F(x)$, or, more explicitly, $y^i = F^i(x_1, \ldots, x_m)$, $i = 1, \ldots, n$, where $F = (F^1, \ldots, F^n)$ is a $C^\infty$ map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$. The definition readily extends to analytic maps between analytic manifolds.

**Definition 1.1.3.** The rank of a map $F : M \to N$ at a point $x \in M$ is defined to be the rank of the $n \times m$ Jacobian matrix $\frac{\partial F^i}{\partial x^j}$ of any local coordinate expression for $F$ at the point $x$. The map $F$ is called regular if its rank is constant.

Standard transformation properties of the Jacobian matrix imply that the definition of rank is independent of the choice of local coordinates. In particular, the set of points where the rank of $F$ is maximal is an open submanifold of the manifold $M$ and the restriction of $F$ to this subset is regular.

**Theorem 1.1.4.** Let $F : M \to N$ be a regular map of rank $r$. Then there exist local coordinates $x = (x_1, \ldots, x_m)$ on $M$ and $y = (y_1, \ldots, y_n)$ on $N$ such that $F$ takes the canonical form

$$y = F(x) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

Thus, all maps of constant rank are locally equivalent and can be linearized by the introduction of appropriate local coordinates. The places where the rank of a map decreases are singularities.
**1.1 Lie Groups**

**Definition 1.1.5.** A set \{f^1, \ldots, f^k\} of smooth real-valued functions on a manifold M having a common domain of definition is called functionally dependent if, for each \(x_0 \in M\), there is neighbourhood U and a smooth function \(H(z_1, \ldots, z_k)\), not identically zero on any subset of \(\mathbb{R}^k\), such that \(H(f^1(x), \ldots, f^k(x)) = 0\) for all \(x \in U\). The functions are called functionally independent if they are not functionally dependent when restricted to any open subset of M.

**Example 1.1.1.** Let us consider the functions

\[
\begin{align*}
f^1(x, y) &= \frac{x}{y} \\
f^2(x, y) &= \frac{xy}{x^2 + y^2}
\end{align*}
\]

are functionally dependent on the upper half plane \(\{y > 0\}\) since the second can be written as a function of the first

\[
f^2 = \frac{f^1}{1 + (f^1)^2}.
\]

For a regular family of functions, the rank tells us how many functionally independent functions it contains.

**Theorem 1.1.6.** If a family of functions \(F\) is regular of rank r, then, in a neighbourhood of any point, there exist r functionally independent functions \(f^1, \ldots, f^r \in F\) with the property that any other function \(f \in F\) can be expressed as

\[
f = H(f^1, \ldots, f^r)
\]

Consequently, if \(f^1, \ldots, f^r\) is a set of functions whose \(m \times r\) Jacobian matrix has maximal rank r at \(x_0\), then, by continuity, they also have rank r in a neighbourhood of \(x_0\), and hence are functionally independent near \(x_0\).

**1.1.3 One-parameter transformation group**

Let \(M\) be an m-dimensional manifold. The collection \(\tau(M)\) of all transformations of the set \(M\) forms a group where the composition of mapping plays the part of the group operation. The group \(\tau(M)\) is called transformation of the set \(M\).

Let now \(G\) be a group. The group homomorphism \(\pi : G \to \tau(M)\) is called a representation of the group \(G\) to \(\tau(M)\). This means that the image \(\pi(g)\) is a transformation of the set \(M\) for every element \(g \in G\) and the following holds

\[
\pi(g_1) \circ \pi(g_2) = \pi(g_1, g_2), \quad \forall g_1, g_2 \in G
\]

The mapping

\[
M \times G \to M; \quad (x, g) \mapsto \pi(g)(x)
\]

is called an operation of the group \(G\) on the set \(M\).

Let \(T : G \to \tau(M)\) a representation of the group \(G\) to \(\tau(M)\). We will consider invertible
transformation $T_g$ in $\mathbb{R}^m$ given by the equation

$$\tilde{x} = T(g, x) =: T_g(x), \quad x, \tilde{x} \in \mathbb{R}^m$$

(1.1)

We assume furthermore that it exist a certain value $g_0$ of the parameter such that the transformation is the identical transformation i.e.

$$x = T_{g_0}(x)$$

and that is does not exist another value close to $g_0$ such that the transformation reduces to the identical transformation.

In short we can say that equation (1.1) defines a single value transformation

$$T_g : \mathbb{R}^m \rightarrow \mathbb{R}^m$$

for all $g$ in a given interval $U \ni g_0$. Then we have a local one-parameter group.

**Definition 1.1.7** (local one-parameter transformation group). A set $G$ of transformation $T_g$ on a manifold $M$ is called one-parameter local transformation group if there exists a subinterval $U_0 \subset U$ containing the identity $e$ such that:

(i) $\exists! e \in U_0$ such that the identity transformation $I = T_e \in G$;

(ii) for any $g \in U_0$, it exist $g^{-1} \in U$ the inverse such that $T_g^{-1} = T_{g^{-1}} \in G$;

(iii) for any $g_1, g_2 \in U_0$ it exists $g_3 \in U$ such that $T_{g_1}T_{g_2} = T_{g_3} \in G$.

**Definition 1.1.8** (Point transformations). A point transformation on a manifold $M$ is a mapping $\tilde{x} = T(x)$ such that $T$ is one-to-one and onto. The transformation corresponding to the inverse of $T$ is denoted by $T^{-1}$.

**Example 1.1.2.** Let us consider the group $G$ of all translations $T_\epsilon$

$$\tilde{x} = T_\epsilon(x) = x + \epsilon, \quad x \in \mathbb{R}, \epsilon \in \mathbb{R}$$

It is easy to verify that

(i)

$$x = T_\epsilon(x) = x + 0;$$

(ii)

$$\tilde{x} = T_{\epsilon_1}^{-1}(x) = T_{-\epsilon_1}(x);$$

(iii)

$$\tilde{x} = T_{\epsilon_1}(T_{\epsilon_2}(x)) = T_{\epsilon_1 + \epsilon_2}(x);$$
1.1 Lie Groups

Example 1.1.3. Let $G$ be the set of the rotation around the origin $\mathbf{R}_\epsilon$ given by

\[
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix} = \mathbf{R}_\epsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix}
cos \epsilon & -\sin \epsilon \\ 
\sin \epsilon & \cos \epsilon
\end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

where $\epsilon \in [0, 2\pi]$.

We can again verify

(i)

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{R}_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}
\]

(ii)

\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \mathbf{R}_{\epsilon_1}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{R}_{-\epsilon_1} \begin{pmatrix} x \\ y \end{pmatrix}
\]

(iii)

\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \mathbf{R}_{\epsilon_1} \mathbf{R}_{\epsilon_2} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{R}_{\epsilon_1 + \epsilon_2} \begin{pmatrix} x \\ y \end{pmatrix}
\]

Transformations in example 1.1.2 and example 1.1.3 are one-parameter local group of transformations. Furthermore example 1.1.3 gives a perfect example of a local group and why we require conditions 1.1.7 to hold close to the identity. In fact in order to guarantee uniqueness of the identity element in the rotation group we have to shrink the interval in which the parameter may vary. At the same time we need that the composition on any two transformation still belongs to the group $G$. It is easy to see that this does necessary hold for the rotation group if the two parameters are sufficiently from from the identity $e$.

Definition 1.1.9. A group $G$ of transformations on a manifold $M$ is said to be continuous, if any two transformations $T_{g_1}, T_{g_2} \in G$ can be connected via a continuous set of elements within the group. In other words, $T_{g_1}$ can be continuously deformed into $T_{g_2}$ within the group $G$.

We will in this thesis consider only continuous transformation groups. In fact Lie groups can be seen concretely as transformation groups which are continuous and analytic in the group operation. The complete understanding of this section will be useful later on the thesis since we will always think of a Lie group as a transformation acting on some manifold $M$.

1.1.4 Lie group of transformations

Definition 1.1.10 (r-parameter Lie group). An r-parameter Lie group is a group $G$ which also carries the structure of a smooth r-dimensional manifold in such a way that both the group operation and the inversion

\[
m : G \times G \to G, \quad m(\epsilon, \delta) = \epsilon \circ \delta, \quad \epsilon, \delta \in G
\]

\[
i : G \to G, \quad i(\epsilon) = \epsilon^{-1}, \quad \epsilon \in G
\]

are smooth maps between manifolds. It is often denoted by $G_r$. 
Therefore a Lie group is a group which is also a differentiable manifold, with the property that the group operation are compatible with the smooth structure. Furthermore, as we will see, Lie groups provide a natural framework for analysing the symmetries of differential equations.

**Example 1.1.4.** Let $G = \mathbb{R}^r$. It is easily seen to be a manifold with a single coordinate chart $U = \mathbb{R}^r$ and $\chi = I$. Let furthermore the group operation be the vector addition $(x, y) \mapsto x + y$. The inversion $x \mapsto -x$ is clearly smooth. Therefore we can say that $\mathbb{R}^r$ is an example of an $r$-parameter abelian Lie group.

**Example 1.1.5.** Let $G = SO(2) \subset GL_2(\mathbb{R})$ the group of rotation in the plane subgroup of the $2 \times 2$ real invertible matrices. We can parametrize $SO(2)$ as

$$SO(2) = \left\{ \begin{pmatrix} \cos \epsilon & -\sin \epsilon \\ \sin \epsilon & \cos \epsilon \end{pmatrix} : 0 \leq \epsilon < 2\pi \right\}$$

We can identify $G$ with the unit circle

$$S^1 = \{ (\cos \epsilon, \sin \epsilon) : 0 \leq \epsilon < 2\pi \}$$

in $\mathbb{R}^2$ that defines the manifold structure on $SO(2)$.

A Lie group homomorphism is a smooth map $\phi : G \to H$ between two Lie groups which respects the group operations:

$$\phi(\epsilon \circ \delta) = \phi(\epsilon) \circ \phi(\delta), \quad \epsilon, \delta \in G$$

If $\phi$ has a smooth inverse, it determines an isomorphism between $G$ and $H$. In practice we will not distinguish between isomorphic Lie groups. For example $(\mathbb{R}^+, \cdot)$ where $\cdot$ denotes the ordinary multiplication is isomorphic to $(\mathbb{R}, +)$. The exponential function $\phi : \mathbb{R} \to \mathbb{R}^+, \phi(t) = e^t$ provides the isomorphism.

We further assume through the thesis the manifold to be connected. Thus unless explicitly otherwise stated all Lie groups are assumed to be connected. Doing so we deliberately exclude discrete symmetries such as reflections focusing on symmetries, like rotations, which can be continuously connected to the identity element. Basically all the infinitesimal techniques that will be introduced in the thesis works only on connected Lie groups.

Often we will be interested not in the full group but only in such elements that are close to the identity. In fact Lie group is homogeneous i.e. every point looks locally like every other point. The neighbourhood of group element $a$ can be mapped into the neighbourhood of group element $b$ by multiplying $a$, and every element in its neighbourhood, on the left by group element $b a^{-1}$ (or on the right by $a^{-1} b$). This maps $a$ into $b$ and points near $a$ into points near $b$. It is therefore necessary to study the neighbourhood of only one group operation in detail. Although geometrically all points are equivalent, algebraically one is special: the identity $e$. It is very useful and convenient to study the neighbourhood of this special group element.

**Definition 1.1.11 (r-parameter local Lie group).** An $r$-parameter local Lie group consists of connected
open subset \( V_0 \subset V \subset \mathbb{R}^r \) containing the origin 0 and smooth maps

\[ m : V \times V \to \mathbb{R}^r \]

defining the group operation and

\[ i : V_0 \to V \]

defining the group inversion such that

\[ \text{(associativity)} \quad x \circ (y \circ z) = (x \circ y) \circ z \text{ for } x, y, z \in V \text{ and } y \circ z, x \circ y \in V; \]

\[ \text{(identity element)} \quad x \circ 0 = x = 0 \circ x \forall x \in V; \]

\[ \text{(inverse)} \quad x \circ x^{-1} = 0 = x^{-1} \circ x \forall x \in V_0. \]

These three axioms read as the normal axioms for a group with the exception that the elements are not necessarily defined everywhere.

**Example 1.1.6.** We present here a non trivial example of a local one-parameter Lie group. Let \( V = \{ x : \lvert x \rvert < 1 \} \subset \mathbb{R} \) with the operation

\[ x \circ y = \frac{2xy - x - y}{xy - 1}, \quad x, y \in V \]

It is easy to verify it is a group. The interesting part is that the inverse map

\[ x^{-1} = \frac{x}{2x - 1}, \quad x \in V \]

is defined for \( V_0 = \{ x : \lvert x \rvert < 1/2 \} \). Therefore we have found a local (but not global!) one-parameter Lie group.

We can actually say that any local Lie group is locally isomorphic to a neighbourhood of the identity of a global Lie group \( G \). Since the only connected one-parameter Lie group is \((\mathbb{R}, +)\) we can say that any local Lie group must coincides with some coordinate charts containing the identity of \( e = 0 \) of \((\mathbb{R}, +)\). Once we know that such a global Lie group does exist we can reconstruct it just from the knowledge of a neighbourhood of the identity of the local Lie group.

**Theorem 1.1.12.** Let \( V_0 \subset V \subset \mathbb{R}^r \) be a local Lie group with group operation \( \circ \) and inversion \( i \). Then there exists a global Lie group \( G \) and a coordinates charts \( \chi : U^* \to V^* \), where \( U^* \) contains the identity element, such that \( V^* \subset V_0 \), \( \chi(e) = 0 \) and

\[ \chi(e \circ \delta) = \chi(e) \circ \chi(\delta), \quad e, \delta \in U^* \]

and

\[ \chi(e^{-1}) = (\chi(e))^{-1}, \quad g \in U^* \]
Moreover there is a unique Lie group $G^*$ having the above properties. If $G$ is any other such Lie group, there exists a covering map $\pi : G^* \to G$ which is a group homomorphism, whereby $G^*$ and $G$ are locally isomorphic Lie group.

The Previous theorem states that it exists a re-parametrization of the Lie group such that the identity element can be take to be $e = 0$ and the composition law of the group to be $+ \text{ i.e. } \epsilon \circ \delta = \epsilon + \delta$. Such a parametrization is of paramount importance since it allow us to treat any transformation as a flow of a vector field, allowing us to determine the infinitesimal generator in such a way that will be seen further in the thesis.

The following gives us an explicit way to find such a parametrization.

**Theorem 1.1.13.** For any Lie group $G$ it exists a re-parametrization such that the new parameter is defined as

$$\tilde{\epsilon} = \int_{\epsilon}^{\epsilon} \frac{ds}{w(s)}, \quad \text{where} \quad w(s) = \frac{\partial(s \circ b)}{\partial b}|_{b=e}$$

In standard literature the new parameter $\tilde{\epsilon}$ is often called canonical parameter. In the proceeding we will avoid to specify the composition law of a given one-parameter Lie group since it can be taken to be the standard addition $+$.  

**Example 1.1.7.** The local Lie group 1.1.6 must coincide with some coordinates charts containing 0 in $\mathbb{R}$.

The identity element of the local group is $e = 0$. Thus if we set

$$w(s) = \frac{\partial^{2xy - x - y}}{\partial y} \bigg|_{y=0} = x^2 - 2x + 1$$

Therefore we get

$$\tilde{\epsilon} = \frac{\epsilon}{\epsilon - 1}$$

Thus if we let

$$\chi : U^* \to V^* \subset \mathbb{R}, \quad \text{where} \quad \chi(\epsilon) = \frac{\epsilon}{\epsilon - 1}, \quad \epsilon \in U^* = \{\epsilon < 1\}$$

we can see that

$$\chi(\epsilon + \delta) = \chi(\epsilon) \circ \chi(\delta) = \frac{2\chi(\epsilon)\chi(\delta) - \chi(\epsilon) - \chi(\delta)}{\chi(\epsilon)\chi(\delta) - 1}$$

$$\chi(\epsilon - \delta) = (\chi(\epsilon))^{-1} = \frac{\chi(\epsilon)}{2\chi(\epsilon) - 1}$$

We can see that such a $\chi$ satisfies the requirements of theorem 1.1.12.

Once we know that such a global Lie group does exist we can reconstruct it just from a neighbourhood of the identity determining the local Lie group.

In practice Lie groups arise most naturally as transformations groups on some manifold $M$. In general a Lie group $G$ will be realized as a group transformation of some manifold if to each group element $\epsilon \in G$ there is associated a map from $M$ to itself.
**Definition 1.1.14** (Local Lie group of transformations). Let $M$ be a manifold and $G$ a local Lie group. A local group of transformation acting on $M$ is given by a local Lie group $G$, an open subset $U$ with

$$\{e\} \times M \subset U \subset G \times M$$

and a smooth map $\Psi : U \to M$ such that

(i) $$\Psi_\epsilon(\Psi_\delta(x)) = \Psi_{\epsilon \circ \delta}(x) \quad (\epsilon, x) \in U, \quad (\epsilon, \Psi_\delta(x)) \in U, \quad (\epsilon \circ \delta, x) \in U; \quad (1.2)$$

(ii) $$\Psi_\epsilon(x) = x, \quad \forall x \in M \text{ and } e \text{ to be the identity of the group } G; \quad (1.3)$$

(iii) $$\Psi_{\epsilon^{-1}}(\Psi_\epsilon(x)) = x \quad (\epsilon, x) \in U, \quad (\epsilon^{-1}, \Psi_\epsilon(x)) \in U; \quad (1.4)$$

The careful reader may now ask where actually lies the difference between one-parameter group of transformations and one-parameter Lie group of transformations. As the name may suggest there are similar. Roughly speaking one-parameter Lie group of transformations are one-parameter group of transformations with in addition few properties. First of all the parameter $\epsilon$ is continuous i.e. it varies in an interval. This allows us to re-parametrize the Lie group considering the identity $e = 0$. Further we require the transformation to be smooth w.r.t. the group operation.

In the literature it is often used the notation $g \circ x$ for the action of a local Lie group. In this thesis we prefer instead the notation $\Psi_\epsilon(x)$ in order to emphasize the that a Lie group can be seen as the flow of a vector field. Equivalently we will use the suggestive notation $\exp(\epsilon v)$ to stress the connection between a Lie group and its infinitesimal generators.

In general, as already said, we will be interested in local Lie group. This implies that the transformations $\exp(\epsilon v)$ may only be defined when the parameter $\epsilon$ is sufficiently close to the identity $e$ which is taken to be 0.

Let us now give few examples to better fix ideas.

**Example 1.1.8.** The group of translation in $\mathbb{R}^m$ already introduced in example 1.1.2. Let $a \neq 0$ be a fixed vector in $\mathbb{R}^m$ and let $G = \mathbb{R}$. Define then

$$\tilde{x} = \Psi_\epsilon(x) = x + \epsilon a, \quad \epsilon \in \mathbb{R}, \quad x \in \mathbb{R}^m$$

This is easily seen to give a global group action.

**Example 1.1.9.** The group $SO(2)$ arises as the group of rotations in the plane $M = \mathbb{R}^2$ given by

$$\left(\begin{array}{c}
\tilde{x} \\
\tilde{y}
\end{array}\right) = R_\epsilon \left(\begin{array}{c}
x \\
y
\end{array}\right) = \left(\begin{array}{cc}
cos \epsilon & -\sin \epsilon \\
\sin \epsilon & \cos \epsilon
\end{array}\right) \left(\begin{array}{c}
x \\
y
\end{array}\right)$$

where $\epsilon \in [0, 2\pi]$. Such a group is a local Lie group of transformations.
Example 1.1.10. The group of scaling. Let $G = \mathbb{R}^+$ be the multiplication group. Fix $m$ real numbers $\alpha_1, \ldots, \alpha_m$ non all zero. Then $\mathbb{R}^+$ acts on $\mathbb{R}^m$ by the scaling transformations

$$
\Psi_\lambda(x) = (\lambda^{\alpha_1}x_1, \ldots, \lambda^{\alpha_m}x_m), \quad \lambda \in \mathbb{R}^+, \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m
$$

This group can be re-parametrized so that the identity element is taken to be $e = 0$ with law of composition given by $\epsilon \circ \delta = \epsilon + \delta + \epsilon \delta$. Let us take in fact $\epsilon = \lambda - 1$ in order to have the transformation

$$
\Psi_\epsilon(x) = ((1 + \epsilon)^{\alpha_1}x_1, \ldots, (1 + \epsilon)^{\alpha_m}x_m), \quad \lambda \in \mathbb{R}^+, \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m
$$

We will in the thesis focus on continuous group. Geometrically a one-parameter group $G$ is continuous in the sense that any point $x \in M$ is carried by the group transformations into the points $\tilde{x} = \Psi_\epsilon(x)$ whose locus is a continuous curve (passing through $x$) and is called a path curve of the group $G$. The group property means that any point of a path curve is carried by $G$ into points of the same curve. The locus of the images $\Psi_\epsilon(x)$ is also termed the $G$-orbit of the point $x$ and denoted $\mathcal{O}$.

See figure 1.1

Definition 1.1.15 (G-orbit). A $G$-orbit of a local transformation group is a minimal non empty group invariant subset of the manifold $M$.

A set $\mathcal{O} \subset M$ is an orbit if it satisfies:

(i) if $x \in \mathcal{O}$, $\epsilon \in G$ and $\exp(\epsilon)(x)$ is defined then $\tilde{x} = \exp(\epsilon)(x) \in \mathcal{O}$;

(ii) if $\tilde{\mathcal{O}} \subset \mathcal{O}$ and $\mathcal{O}$ satisfies part (i) then either $\tilde{\mathcal{O}} = \mathcal{O}$ or $\tilde{\mathcal{O}}$ is empty.

Figure 1.1: The $G$-orbit of the rotation group
1.1 Lie Groups

1.1.5 Vector Field and Flow

This section is of paramount importance to a complete understanding of the concept of Lie group of transformations and nevertheless the main reason why Lie groups are so widely used.

The study of Lie groups would simplify greatly if the group composition law could somehow be linearised, and this linearisation retained a substantial part of the information inherent in the original group composition law. This in fact can be done. Lie algebras are constructed by linearising Lie groups. A Lie group can be linearised in the neighbourhood of any of its points, or group operations. Linearisation amounts to Taylor series expansion about the coordinates that define the group operation. What is being Taylor expanded is the group composition function. This function can be expanded in the neighbourhoods of any group operations.

Let us suppose that $C$ is a smooth curve on a manifold $M$ parametrized by

$$\phi : I \rightarrow M$$

where $\{0\} \ni I \subset \mathbb{R}$

At each point $x = \phi(\epsilon) \in C$ the curve has a tangent vector

$$\dot{\phi}(\epsilon) = \frac{d\phi}{d\epsilon} = \left(\dot{\phi}^1(\epsilon), \ldots, \dot{\phi}^m(\epsilon)\right).$$

We will then denote the tangent vector to $C$ at $x$ with

$$v_{|x} = \dot{\phi}(\epsilon) = \dot{\phi}^1(\epsilon) \frac{\partial}{\partial x_1} + \cdots + \dot{\phi}^m(\epsilon) \frac{\partial}{\partial x_m}$$

Example 1.1.11. The helix

$$\phi(\epsilon) = (\cos \epsilon, \sin \epsilon)$$

in $\mathbb{R}^3$, with coordinates $(x, y, z)$, has the tangent vector

$$v_{|x} = \dot{\phi}(\epsilon) = -\sin \epsilon \frac{\partial}{\partial x} + \cos \epsilon \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

See figure 1.2.

Two curves

$$C = \{\phi(\epsilon)\}, \quad \tilde{C} = \{\tilde{\phi}(\theta)\}$$

passing through the same point

$$x = \phi(\epsilon^*) = \tilde{\phi}(\theta^*)$$

have the same tangent vector iff their derivatives agree

$$\frac{d\phi}{d\epsilon}(\epsilon^*) = \frac{d\tilde{\phi}}{d\theta}(\theta^*) \quad (1.5)$$

It is easily seen how this concept is independent of the local coordinate system. Let $x = \phi(\epsilon) = $
Figure 1.2: Vector field and integral curves

\((\phi^1(\epsilon), \ldots, \phi^m(\epsilon))\) be the local coordinate expression of \(x = (x_1, \ldots, x_m)\) and let \(y = \psi(x)\) a diffeomorphism. Then \(y = \psi(\phi(\epsilon))\) is the local coordinate formula for the curve in terms of the y-coordinates. Thus equation (1.5) holds iff

\[
\frac{d}{d\epsilon} \psi(\phi(\epsilon^*)) = \frac{d}{d\theta} \psi(\hat{\phi}(\theta^*))
\]

The tangent vector \(v|_x = \hat{\phi}(\epsilon)\) in x-coordinates takes the form

\[
v|_{y=\psi(x)} = \sum_j \frac{d}{d\epsilon} \psi^j(\phi(\epsilon)) \frac{\partial}{\partial y^j} = \sum_j \sum_i \frac{\partial \psi^j}{\partial x^i}(\phi(\epsilon)) \frac{d \phi^k}{d\epsilon} \frac{\partial}{\partial y^j}
\]  \hspace{1cm} (1.6)

in the y-coordinates.

The collection of all tangent vectors to all possible curves passing through a given point \(x \in M\) is called the tangent space to \(M\) at \(x\) and it will be denoted by \(TM|_x\). Obviously if \(M\) is m-dimensional so \(TM|_x\) is a \(m\)–dimensional vector space spanned by \(\{\partial_{x^1}, \ldots, \partial_{x^m}\}\) provides a basis for the space. The collection of all tangent spaces corresponding to all points \(x \in M\) is called the tangent bundle of \(M\) denoted by \(TM = \bigcup_{x \in M} TM|_x\).

A vector field \(v\) on \(M\) is a smoothly varying assignment of tangent vectors on \(v|_x \in TM|_x\) to each point \(x \in M\). In terms of local coordinates the vector field has the form

\[
\left. v \right|_x = \xi^1(x) \frac{\partial}{\partial x^1} + \cdots + \xi^m(x) \frac{\partial}{\partial x^m}
\]  \hspace{1cm} (1.7)

where each \(\xi^i(x)\) is an analytic function of \(x\) so that we deal exclusively with analytic vector field. The previous definition of vector field will be widely used in the thesis.

**Definition 1.1.16 (Integral curve).** An integral curve of a vector field \(v\) is a smooth curve \(x = \phi(\epsilon)\)
whose tangent vector at any point coincides with the value of \( v \) at the same point

\[
\dot{\varphi}(\epsilon) = v|_{\varphi(\epsilon)}, \quad \forall \epsilon \in I
\]

In local coordinates \( x = \varphi(\epsilon) = (\varphi_1(\epsilon), \ldots, \varphi_m(\epsilon)) \) must be a solution of the autonomous system of ordinary differential equations

\[
\begin{cases}
\frac{dx_i}{d\epsilon} = \xi_i(x), & i = 1, \ldots, m \\
\varphi(0) = x_0
\end{cases}
\] (1.8)

Different integral curves through a given point can have different domains of definitions \( I \) and the unique one corresponding to the maximum domain of definition is called the maximal integral curve through the point \( x \).

**Definition 1.1.17** (Flow). If \( v \) is a vector field we can denote by \( \Psi(x; \epsilon) \) the flow generated by \( v \) the parametrized maximal integral curve passing through \( x \). The flow of a vector field will have the three following properties

(i)

\[
\Psi(\Psi(x; \epsilon); \delta) = \Psi(x; \delta + \epsilon);
\] (1.9)

(ii)

\[
\Psi(x; 0) = x;
\] (1.10)

(iii)

\[
\frac{d}{d\epsilon}\Psi(x; \epsilon) = v|_{\Psi(x; \epsilon)}.
\] (1.11)

Equation (1.11) simply states that \( v \) is tangent to the curve \( \Psi(x; \epsilon) \) for a fixed \( x \).

The vector field \( v \) is called the infinitesimal generator of the action.

Comparing now (1.9) and (1.10) with (1.2) and (1.3) we can see that the flow generated by a vector field is the same as a local group action of the Lie group \( G = (\mathbb{R}, +) \) acting on a manifold \( M \).

Thanks to theorem 1.1.12 we can thus always find a new coordinates chart such that any local Lie group of transformations has identity element \( e = 0 \) and composition law given by \( \epsilon \circ \delta = \epsilon + \delta \). This means that any local Lie group can be seen as the action of a flow of a vector field and furthermore the infinitesimal generator evaluated at \( x \in M \) of the flow, defined as in (1.7), gives the infinitesimal transformation of \( x \).

**Definition 1.1.18** (Infinitesimal generator). Let us consider a one-parameter Lie group of transformations \( \exp(\epsilon v) \) acting on a manifold \( M \) associated to the vector field

\[
v = \xi(x) \cdot \nabla = \xi^1(x) \frac{\partial}{\partial x_1} + \cdots + \xi^m(x) \frac{\partial}{\partial x_m}
\] (1.12)

Each \( \xi^i \) is called the infinitesimal of \( x_i \) and the corresponding vector field \( v \) is called the infinitesimal generator of \( \exp(\epsilon v) \)
The following theorem is of paramount importance since it asserts that one-parameter local groups are determined by their infinitesimal transformations.

**Theorem 1.1.19 (Lie).** Let $G$ be a one-parameter local group of transformations. The one-parameter family of transformations $\tilde{x} = \Psi(\epsilon)(x)$ associated with an analytic vector field $v$ of the form (1.12) is the unique solution to the autonomous system of ordinary differential equations (known as the Lie’s equations)

$$\begin{cases} \frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}) \\ \tilde{x}|_{\epsilon=0} = x \end{cases}$$

(1.13)

where the $\xi^i(x)$ are the coefficients of $v$ at $x$.

System (1.13) suggests suggestive notation

$$\exp(\epsilon v)x \equiv \Psi(\epsilon)(x)$$

for a *local Lie group* of transformations.

**Theorem 1.1.20 (Lie).** Let $v$ be a vector field. Solutions of the problem (1.13) generates the local one-parameter Lie group for which the vector field $v$ is its tangent vector field.

We can imagine the *one-parameter group* to act component wise i.e.

$$(\tilde{x}_1, \ldots, \tilde{x}_m) = \tilde{x} = \Psi(\epsilon)(x) = (X^1_\epsilon(x), \ldots, X^m_\epsilon(x))$$

for some $X_\epsilon$ smooth functions.

We will assume that one-parameter group $G$ of transformations is given in a *canonical parameter* and expand the functions $X^i_\epsilon$ into the Taylor series in a neighbourhood of the identity $\epsilon = 0$. We thus obtain the infinitesimal transformation of the group $G$

$$\tilde{x}_i = x_i + \epsilon \xi^i(x) + O(\epsilon^2), \quad i = 1, \ldots, m$$

where

$$\left. \frac{dX^i_\epsilon(x)}{d\epsilon} \right|_{\epsilon=0} = \xi^i(x),$$

Geometrically the infinitesimal transformation of a *Lie group* $G$ defines the tangent vector

$$\xi(x) = (\xi^1(x), \ldots, \xi^m(x))$$

at the point $x$ to the $G$-orbit, where the orbit of the one-parameter group action are the maximal integral curves of the vector field $v$. For some deeper notion on $G$-orbit we refer to section A.1.1. Therefore $\xi$ is called the tangent vector field of the group $G$. The tangent vector is often written in local coordinates as

$$v = \sum_i \xi^i(x) \frac{\partial}{\partial x_i}$$
Theorem 1.1.21. The one-parameter group of transformations \( \tilde{x} = \exp(\epsilon \nu)(x) \) is equivalent to

\[
\tilde{x} = \Psi_\epsilon(x) = \exp(\epsilon \nu)x = x + \epsilon \nu x + \frac{\epsilon^2}{2} \nu^2 x + \ldots \\
= \left( 1 + \epsilon \nu + \frac{\epsilon^2}{2} \nu^2 + \ldots \right) x = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \nu^k x
\]

(1.14)

where \( \nu \) is defined in (1.7).

Here is how flows of vector fields give rise to the transformations on the underlying space. Let \( \exp(\epsilon \nu)(x) \) be the transformation. Then for each fixed \( \epsilon \), \( \exp(\epsilon \nu)(x) : M \to M \) defines a transformation on \( M \). Different \( \epsilon \) leads to a different transformation \( \exp(\epsilon \nu) \) and the set of all these is called one-parameter family of transformations. To better understand the analogy between flow and transformation let us think at the vector field as the surface velocity of a river (which would be the manifold \( M \) in our analogy). Let us now imagine to drop a particle in such a river and let it flows for a given time \( \epsilon \). The final location of such a particle is the transformation \( \exp(\epsilon \nu) \). In particular different values of the parameter \( \epsilon \) lead to different transformation \( \exp(\epsilon \nu) \) and the set of all of these transformations give rise to a one-parameter group of transformations.

Remark 1.1.22. Uniqueness of solutions of (1.13) guarantees us that the flow generated by the vector field \( \nu \) coincides with the local action of \( G = \mathbb{R} \) on \( M \). The transformation \( \exp(\epsilon \nu) \) is referred in standard literature as exponentiating the vector field \( \nu \) and represented by the notation

\[ \tilde{x} = \exp(\epsilon \nu)x \equiv \Psi_\epsilon(x) \]

Summarizing the connection between flows of vector fields and local one-parameter Lie group of transformations we can say that \( \nu \) is the infinitesimal generator of the transformation. The orbits of the one-parameter group action are the maximal integral curves of the vector field \( \nu \). Conversely if \( \exp(\epsilon \nu)(x) \) is any one-parameter group of transformations acting on \( M \) then its infinitesimal generator is obtained by specializing (1.11) at \( \epsilon = 0 \)

\[ \nu|_x = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon \nu)(x) \]

Furthermore there are two ways to explicitly find a one-parameter family of transformations:

(i) express the group in terms of a power series (1.14) developed from the infinitesimal generator;

(ii) solve an initial value problem (IVP). In particular we can restate (1.11) as

\[ \frac{d}{d\epsilon} [\exp(\epsilon \nu)x] = \nu|_{\exp(\epsilon \nu)x} \]

(1.15)

We will then evaluate \( \nu|_x \) computing (1.15) at \( \epsilon = 0 \).
Example 1.1.12. Let \( M = \mathbb{R} \) with coordinates \( x \) and consider the vector field

\[
v = \frac{\partial}{\partial x}
\]

Then

\[
\Psi_\epsilon(x) = \exp(\epsilon v)x = \exp(\epsilon \partial_x)x = x + \epsilon
\]

which is globally defined. For the vector field \( x \partial_x \) we recover the exponential

\[
\exp(x \partial_x)x = e^x
\]

since it must be the solution of the ordinary differential equation

\[
\begin{aligned}
&\dot{x} = x; \\
&x(0) = x;
\end{aligned}
\]

\hspace{1cm} (1.16)

![Figure 1.3: Vector field of the translation group](image)

Example 1.1.13. Let us now consider the group of rotations in the plane

\[
\Psi_\epsilon(x, y) = \exp(\epsilon v)(x, y) = (x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon)
\]

Its infinitesimal generator is given by

\[
\begin{aligned}
&\xi(x, y) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (x \cos \epsilon - y \sin \epsilon) = -y \\
&\nu(x, y) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (x \sin \epsilon + y \cos \epsilon) = x
\end{aligned}
\]

\hspace{1cm} (1.17)
Thus the infinitesimal generator has the form

\[ v = -y \partial_x + x \partial_y \]

It can be easily checked that it agrees with the solution of the system

\[ \begin{cases} \frac{d\tilde{x}}{d\epsilon} = -y \\ \frac{d\tilde{y}}{d\epsilon} = x \end{cases} \]  

(1.18)

![Figure 1.4: Vector field of the rotation group](image)

The effect of a change of coordinates \( y = \psi(x) \) on a vector field \( v \) is determined by its effect on each individual tangent vector \( v|_{\tilde{x}}, \tilde{x} \in M \) as given by (1.6). Thus if \( v \) is a vector field whose expression is

\[ v = \sum_i \xi^i(x) \frac{\partial}{\partial x_i} \]

and \( y = \psi(x) \) is a change of coordinates, then \( v \) has the formula

\[ v = \sum_j \sum_i \xi^i(\psi^{-1}(y)) \frac{\partial \psi^j}{\partial x_i} \left( \frac{\partial}{\partial y_j} \right) \]

in the y-coordinates.

**Definition 1.1.23.** A change of coordinates \( y = \psi(x) \) defines a set of canonical coordinates for the one-parameter Lie group of transformation

\[ \tilde{x} = \exp(\epsilon v)(x) \]  

(1.19)
if in term of such coordinates the group becomes

\[
\begin{aligned}
\tilde{y}_i &= y_i, \quad i = 1, \ldots, n - 1 \\
\tilde{y}_n &= y_n + \epsilon
\end{aligned}
\]  

(1.20)

**Theorem 1.1.24.** For any local Lie group of transformations there exists a set of canonical coordinates \( y = \psi(x) \) such that (1.19) is equivalent to (1.20).

The next result shows how by suitably choosing local coordinates we can simplify the expression for objects on manifold.

**Proposition 1.1.25.** Let us suppose \( v \) is a vector field not vanishing at \( x_0 \in M \). Then there is a local coordinate chart \( y = (y_1, \ldots, y_m) \) at \( x_0 \) such that in term of this coordinates \( v = \frac{\partial}{\partial y_1} \).

The importance of canonical coordinates will be seen in practice in section 1.3.

**Remark 1.1.26.** We would like to stress the difference between canonical parameters and canonical coordinates. They are both a change of coordinates charts. The difference is that the former is a change of coordinates of the parameters \( \epsilon \) such that the composition law in +. Further this is needed if we are to consider any local Lie group as the flow of a vector field. The latter is a change of coordinates charts on the manifold \( M \) and it is one of the basic methods for solving ordinary differential equations with known symmetries.

As a proof of the fact let us think to the standard example of the rotation group. It is already written in a canonical parameters \( \epsilon \). Let us think of example 1.1.3 where the vector field has been found. But it is not written in canonical variables. In fact we can reparametrize the group such that the rotation group can be seen as a translation group after a suitable change of variables. It will be seen in 1.3.3.

### 1.1.6 Lie brackets and Lie algebras

The most important operator on vector fields is their Lie brackets.

**Definition 1.1.27** (Lie bracket). Let us consider an \( r \)-parameter Lie groups of transformations with infinitesimal generators \( \{v_j\}_{j=1,\ldots,r} \) defined

\[
v_j = \sum_i \xi^j_i(x) \frac{\partial}{\partial x_i}
\]

(1.21)

on a manifold \( M \). Their Lie bracket \([v_j, v_k]\) is the unique vector field satisfying

\[
[v_j, v_k](f) = v_j (v_k (f)) - v_k (v_j (f))
\]

(1.22)

for any smooth function \( f : M \to \mathbb{R} \).
In particular given two infinitesimal generators of the form
\[ v = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x_i}, \quad w = \sum_{i=1}^{m} \nu^i \frac{\partial}{\partial x_i} \]
then Lie bracket of \( v \) and \( w \) is another first order operator
\[ [v, w] = \sum_{i=1}^{m} [v(\nu^i) - w(\xi^i)] \frac{\partial}{\partial x_i} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \xi^i \frac{\partial \nu^j}{\partial x_j} - \nu^j \frac{\partial \xi^i}{\partial x_j} \right] \frac{\partial}{\partial x_i} \]  
(1.23)

Proposition 1.1.28. The Lie bracket has the following properties:

(bilinearity)
\[ [c_i v_i + c_j v_j, v_k] = c_i [v_i, v_k] + c_j [v_j, v_k], \]
\[ [v_k, c_i v_i + c_j v_j] = c_i [v_k, v_i] + c_j [v_k, v_j]. \]

with \( c_i \) and \( c_j \) any constants;

(skew-symmetry)
\[ [v_i, v_j] = -[v_j, v_i]; \]

(Jacobi identity)
\[ [v_i, [v_j, v_l]] + [v_j, [v_l, v_i]] + [v_l, [v_i, v_j]] = 0 \]

Example 1.1.14. The Lie bracket of the scaling and translation vector fields on \( M = \mathbb{R} \) is
\[ [\partial_x, x \partial_x] = \partial_x \]
reflecting the non-commutativity of the operations of translation and scaling. In particular, if the two flows commute, then the Lie bracket of their infinitesimal generators is necessarily zero by theorem 1.1.36.

Once we have introduced the Lie bracket operator we can now define the main algebraic structure of Lie group, the Lie algebra. Such a structure is of paramount importance since it is, in a certain sense, the infinitesimal generator of the Lie group \( G \).

It will be seen in next section that the set of all solutions of the determining equations for a given differential equation is a vector space closed with respect to the Lie bracket of infinitesimal generators. Such vector spaces are called Lie algebras. This section is devoted to this particular vector spaces. They are of great importance since they are the link between one-parameter Lie group of transformations and the infinitesimal generator.

The study of a Lie group \( G \) can in fact be greatly simplified by considering the so called tangent space \( V \) of \( G \) around the identity of the group. This tangent space \( V \) is the space generated by the infinitesimal generator \( v \). We have introduced an operation on this vector field so that the resulting algebraic structure, called the Lie algebra, determines the local structure of a group. Actually the
tangent space itself is the Lie algebra. Let us notice that makes perfectly sense to talk about tangent space since a Lie group is defined together with a manifold.

Let us consider an r-parameter Lie group of transformations. The group under consideration has an infinite number of elements. However the properties of the group, as already mentioned, may be deduced from a finite number r of operators, called the infinitesimal generators.

**Definition 1.1.29.** Let $G$ be a group acting on a manifold $M$. A vector field $v$ on $M$ is called $G$-invariant if it is unchanged by the action of any group element $g \in G$.

Associated to any Lie group, there are two different Lie algebras, which we denote by $g_L$ and $g_R$ respectively. Both Lie algebras play a role, although the right-invariant one is by far the more useful of the two, due to the convention that Lie groups act on the left on manifolds. Therefore, in the sequel, when we talk about the Lie algebra associated with a Lie group, we shall mean the right-invariant Lie algebra, and denote it by $g = g_R$. For a mild treatment of the topic the interested reader can see A.1.3, for a deeper treatment we refer instead to [Olv93].

**Definition 1.1.30** (Lie algebra). A Lie algebra $g$ is a vector space over some field $\mathcal{F}$ with an additional law of combination of elements (the Lie bracket)

$$[\cdot, \cdot] : g \times g \to g$$

such that $\forall a, b \in \mathcal{F}$ and $\forall v_i, v_j, v_k \in g$ the following properties hold:

(Closure) $[v_i, v_j] \in g$;

(Bilinearity) $[v_i, av_j + bv_k] = a[v_i, v_j] + b[v_i, v_k]$;

(Anticommutativity) $[v_i, v_j] = -[v_j, v_i]$;

(Jacobi identity) $[v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0$;

A Lie algebra $g$ spanned by r vector fields is r-dimensional. It is often denoted by $g_r$. We will in general avoid this notation if it does not create confusion.

**Theorem 1.1.31.** Let us consider a set of infinitesimal generators $\{v_j\}_{j=1,\ldots,r}$ of an r-parameter Lie group of transformations $G$. Therefore the set $\{v_j\}_{j=1,\ldots,r}$ form an r-dimensional Lie algebra $g^r$.

**Theorem 1.1.32.** The Lie bracket of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator, in particular

$$[v_i, v_j] = C^h_{ij} v_h$$

where the coefficients $C^h_{ij}$ are constants called structure constants.
The structure of the Lie algebra is completely determined by its structure constants.

**Definition 1.1.33.** Equation (1.24) is called the commutation relation of the r-parameter Lie group of transformations with infinitesimal generator (1.21).

**Theorem 1.1.34.** The structure constants defined in (1.24) satisfy the relations

\[ C^h_{ij} = -C^h_{ji} \]

\[ C^h_{ij} C^d_{bg} + C^h_{jg} C^d_{hi} + C^h_{gi} C^d_{hj} = 0 \]

There is a more geometric characterization of the Lie bracket of two vector fields as the infinitesimal commutator of the two one-parameter group \( \exp(\epsilon v_i) \) and \( \exp(\epsilon v_j) \).

**Theorem 1.1.35.** Let \( v_i \) and \( v_j \) be smooth vector fields on a manifold \( M \). For each \( x \in M \) the commutator

\[ \Psi_\epsilon(x) = \exp(-\sqrt{\epsilon}v_j) \exp(-\sqrt{\epsilon}v_i) \exp(\sqrt{\epsilon}v_j) \exp(\sqrt{\epsilon}v_i) \]

defines a smooth curve for sufficiently small \( \epsilon \geq 0 \). The Lie bracket \( [v_i, v_j]_x \) is the tangent vector to this curve at the end point \( \Psi_0(x) = x \)

\[ [v_i, v_j]_x = \frac{d}{d\epsilon} \bigg|_{\epsilon=0^+} \Psi_\epsilon(x) \]

![Figure 1.5: Commutator construction of Lie bracket](image)

The previous theorem may seem at first sight useless as much as abstruse. As most of the theory it makes sense when one thinks to the action of the Lie group of transformation as the action of a flow on a vector fields. Theorem 1.1.35 means that if we take a point \( x \in M \) and we apply \( \exp(\sqrt{\epsilon}v_i) \) we end up on a point \( \tilde{x} \in M \). Applying now in the order \( \exp(\sqrt{\epsilon}v_j), \exp(-\sqrt{\epsilon}v_i) \) and \( \exp(-\sqrt{\epsilon}v_j) \) we
thus get a point \( x^* \in M \). If we now take the Lie bracket \([v_i, v_j]\) (let us remember that the Lie bracket is a vector field by definition) and we follow the integral curve tangent to the vector \([v_i, v_j]\) and we stop “at \( \epsilon \)” we would obtain the same point \( x^* \in M \) as before. This can be easily seen in figure 1.5.

**Theorem 1.1.36.** Let \( v_i, v_j \) be two vector field on \( M \). Then

\[
\exp(\epsilon v_i) \exp(\delta v_j) x = \exp(\delta v_j) \exp(\epsilon v_i) x
\]

for all \( \epsilon, \delta \in \mathbb{R}, x \in M \) if and only if

\[
[v_i, v_j] = 0
\]

**Definition 1.1.37.** A subspace \( h \subset g \) is called sub-algebra of the Lie algebra \( g \) if for any \( v_i, v_j \in h \), \([v_i, v_j] \in h\).

The most convenient way to display the structure of a give Lie algebra is the commutator table.

**Definition 1.1.38 (Commutator tables).** Let us consider again an \( r \)-parameter Lie groups of transformations with infinitesimal generators \( \{v_j\}_{j=1,\ldots,r} \) defined as in (1.21) on a manifold \( M \). The commutator table for the algebra will be the \( r \times r \) table whose \((i, j)\)-entry is \([v_i, v_j]\).

Let us notice that the table is skew-symmetric with all zeros on the main diagonal. Further the structure constants can be easily read. In fact \( C_{i,j}^h \) is the coefficient of \( v_h \) in the \((i, j)\)-entry.

**Example 1.1.15.** Let us consider \( \mathfrak{sl}(2) \) the Lie algebra of special linear group \( SL(2) \) of \( 2 \times 2 \) matrices with trace 0. We will take the basis

\[
A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

The commutator table is therefore

<table>
<thead>
<tr>
<th></th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>0</td>
<td>( A_1 )</td>
<td>(-2A_2)</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>(-A_1)</td>
<td>0</td>
<td>( A_3 )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( 2A_2 )</td>
<td>(-A_3)</td>
<td>0</td>
</tr>
</tbody>
</table>

Briefly speaking in the thesis we will mainly concern with the infinitesimal description of the action of a Lie group \(^1 \) \( G \) on a manifold \( M \). Let \( g \) be the Lie algebra of the Lie group \( G \). Then for any \( v \in g \) the one-parameter group \( \epsilon \rightarrow \exp(\epsilon v) \) acts on \( G \). We can thus introduce the vector field \( v \in \mathfrak{X}(M) \) whose integral curves are of the form \( \epsilon \rightarrow \exp(\epsilon v)x \). The first fundamental theorem of Lie asserts that the action of \( G \) on \( M \) is described by a homomorphism of \( G \) into the group of all analytic diffeomorphism of \( M \). Thus we have the analogy

\(^1\)we will use equivalently the terminology Lie group and Lie group of transformations.
1.2 Lie Group of Transformations for PDE’s

The Lie symmetries of differential equations naturally form a group. Such a group, as we have seen, is called Lie group and it is a point invertible transformations of both dependent and independent variables of the PDE.

Of course in order to use the symmetry group in practice we first need to find the symmetries of the equation.

The first method used was to make a general change of all variables and then to force the new variables to satisfy the original equation. This method leads to a complicated non-linear system of differential equations for the transformation used. Lie proved that such approach is unnecessary. He was able in fact to develop an efficient method based on an infinitesimal formulation of the problem. He reduced therefore the problem to a solution of a linear system of partial differential equations. The solution of the determining equations will give us the symmetry transformation.

The present section is devoted to such a method developed by Lie. We will in the first section focus on system of algebraic equations and then pass to the more complicated case of differential equations.

From now on we will assume the standard multi-index notation. We will consider a PDE of the form

\[ F(x, \{D^\alpha\}_{|\alpha| \leq m}) = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha \]  

(1.25)

with \( x \in \mathbb{R}^N, \alpha = (\alpha_1, \ldots, \alpha_N) \) a N-tuple of integers numbers, \( |\alpha| = \alpha_1 + \cdots + \alpha_N \) the length of \( \alpha \) and \( D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} \).

Anyway in the proceeding of the thesis we will refer to a general PDE of the form (1.25) as

\[ F(x, u^{(m)}) = 0 \]

meaning with \( u^{(m)} \) the dependent variable with derivatives up to order \( m \). We chose such a notation both to conform ourself to the existing literature and since it is the most intuitive and less heavy notation.

Further, if not specified, all definitions are as in the previous section. Moreover we will always refer to a group element \( \epsilon \in G \). From now on any manifold is consider to be \( N \)-dimensional.

1.2.1 Invariant Subsets and Equations

Let \( G \) be a group of transformations acting on the manifold \( M \). A subset \( S \subset M \) is called \( G \)-invariant if it is unchanged by the group transformations, meaning

\[ \exp(\epsilon v)(x) \in S \quad \text{whenever} \quad \epsilon \in G \quad \text{and} \quad x \in S \]

The most important classes of invariant subsets are the varieties defined by the vanishing of one or more functions.

An invariant of a transformation group is defined as a real-valued function whose values are
unaffected by the group transformations. The determination of a complete set of invariants of a given
group action is a problem of supreme importance for the study of equivalence and canonical forms.
In the regular case, the orbits, and hence the canonical forms, for a group action are completely
characterized by its invariants.

**Definition 1.2.1.** Let $G$ be a local one-parameter group of transformations acting on a manifold $M$.
A subset $S \subset M$ is called $G$–invariant and $G$ is called symmetry group of $S$ if whenever $x \in S$ and
$\epsilon \in G$ thus $\exp(\epsilon v)(x)$ is defined and $\exp(\epsilon v)(x) \in S$.

**Definition 1.2.2 (Invariant function).** Let $G$ be a Lie group acting on a manifold $M$. An invariant of
$G$ is a real-valued function $f : M \to \mathbb{R}$ which satisfies for all

$$f(\tilde{x}) = f(\exp(\epsilon v)(x)) = f(x), \quad \forall \epsilon \in G$$

**Proposition 1.2.3.** Let $f : M \to \mathbb{R}$. The following conditions are equivalent:

(i) $f$ is a $G$–invariant function;

(ii) $f$ is constant on the orbit of $G$;

(iii) the level sets $\{f(x) = c\}$ of $f$ are $G$–invariant subset of $M$.

**Example 1.2.1.** Let us consider the rotation group 1.1.3 acting on $\mathbb{R}^2$. Thus the orbits are clearly
the spheres $r = |x| = constant$, and hence any orthogonal invariant is a function of the radius:

$$f = F(r).$$

A fundamental problem is the determination of all the invariants of a group of transforma-
tions. Note that if $f^1(x), \ldots, f^k(x)$ are invariants, and $H(z_1, \ldots, z_k)$ is any function, then $f(x) =
H(f^1(x), \ldots, f^k(x))$ is also invariant. Therefore, we need only find a complete set of functionally
independent invariants having the property that any other invariant can be written as a function of
these fundamental invariants.

It can be proved that the number of independent local invariants of a regular transformation group
is completely determined by the orbit dimension. Indeed the following theorem holds.

**Theorem 1.2.4.** Let $G$ be a Lie group acting on the $m$–dimensional manifold $M$ with $s$–dimensional
orbits. At each $x \in M$, there exist $m - s$ functionally independent local invariants $f^1, \ldots, f^{m-s}$,
defined on a neighbourhood $U$ of $x$, with the property that any other local invariant $f$ defined on $U$
可以 be written as a function of the fundamental invariants:

$$f = H(f^1, \ldots, f^{m-s}).$$

Moreover, in the regular case, two points $y, z \in U$ lie in the same orbit of $G$ if and only if the
invariants all have the same value
\[ f^i(y) = f^i(z), \quad i = 1, \ldots, m - s \]

Theorem 1.2.4 provides a complete answer to the question of local invariants of group actions.

**Proposition 1.2.5.** Let \( G \) be a Lie group of transformations acting on a manifold \( M \) with infinitesimal generator \( v \). A function \( f : M \to \mathbb{R} \) is invariant under \( G \) if and only if
\[ v[f] = 0 \]
for all \( x \in M \) and every infinitesimal generator \( v \in g \).

Thus the invariants \( u = f(x) \) of a one-parameter group with infinitesimal generator
\[ v = \sum_{i=1}^{N} \xi^i(x) \frac{\partial}{\partial x_i} \]
has to satisfy the first order homogeneous partial differential equation
\[ \sum_{i=1}^{N} \xi^i(x) \frac{\partial f(x)}{\partial x_i} = 0 \quad (1.26) \]
The solutions of (1.26) are effectively found by the method of characteristics. We replace the partial differential equation by the characteristic system of ordinary differential
\[ \frac{dx_1}{d\xi^1(x)} = \cdots = \frac{dx_N}{d\xi^N(x)} \]
The general solution can be written in the form
\[ f_1(x) = c_1, \ldots, f_{N-1}(x) = c_{N-1} \]
where the \( c_i \) are constants of integration. The resulting functions \( f_1, \ldots, f_{N-1} \) form a complete set of functionally independent invariants of the one-parameter group generated by \( v \).

We can extend the procedure to a system of algebraic equations. Let us now consider a system of algebraic equations of the form
\[ F_\nu(x) = 0, \quad \nu = 1, \ldots, l \quad (1.27) \]
in which \( F \) are smooth functions defined for \( x \) on some manifold. A solution is therefore a point \( x \in M \) such that \( F(x) = 0 \) \( \forall \nu \)
\[ S_F = \{ x \in M : F_\nu(x) = 0, \ \forall \nu \} \subset M \]

**Definition 1.2.6.** A symmetry of a system of equations of the form (1.27) is a transformation \( \exp(\epsilon v) \) mapping any solution of the system to another solution i.e. \( \exp(\epsilon v)(S_F) \subset S_F \).
Thus a group $G$ is called symmetry group of the system if and only if the variety $\mathcal{S}_F$ is a $G$-invariant subset of $M$.

Therefore, a symmetry group of a system of equations maps solutions to other solutions: if $x \in M$ satisfies (1.27) and $\epsilon \in G$ is any group element such that $\exp(\epsilon v)(x)$ is defined, then the transformed point $\tilde{x}$ is also a solution to the system. Knowledge of a symmetry group of a system of equations allows us to construct new solutions from old ones.

As it has already been mentioned before, the real strength of Lie group theory is that they can be studied just looking the infinitesimal generator of the group action. This provides the key to the determination of the symmetry groups of a system of differential equations.

**Proposition 1.2.7.** Let $G$ be a group acting on a manifold $M$ and $F : M \to \mathbb{R}^l$ a smooth function. Then $F$ is a $G$-invariant function iff every level set $\{F(x) = c\}$, $c \in \mathbb{R}^l$ is a $G$-invariant subset of of $M$.

**Theorem 1.2.8.** Let $G$ a local Lie group of transformations acting on the $N$-dimensional manifold $M$. Let $F : M \to \mathbb{R}^l$ with $l \leq N$, define a system of algebraic equations

$$F_\nu(x) = 0 \quad \nu = 1, \ldots, l$$

and assume the system is of maximal rank i.e. the Jacobian matrix is of rank $l$ at any solution $x$. Then $G$ is a symmetry group of the system iff

$$\nu[F_\nu(x)] = 0 \quad \nu = 1, \ldots, l \quad \text{whenever} \quad F(x) = 0$$

(1.28)

for every infinitesimal generator $v \in g$.

**Example 1.2.2.** Let us consider the group of rotation in the plane $G = SO(2)$ with infinitesimal generator given by

$$v = -y \partial_x + x \partial_y$$

The unit circle

$$S^1 = \{x^2 + y^2 = 1\}$$

is an invariant subset of $G$ as given

$$\zeta(x, y) = x^2 + y^2 - 1$$

it easily follows

$$\nu[\zeta] = -2xy + 2xy = 0.$$ 

Further the maximal rank condition does hold since the gradient

$$\nabla \zeta = (2x, 2y)$$
1.2 Lie Group of Transformations for PDE’s

(a) The function $F(x, y)$

(b) Level curves

Figure 1.6

does not vanish on $S^1$. A less trivial example is given by the function

$$F(x, y) = x^4 + x^2 y^2 + y^2 - 1$$

It is easy to see that (1.28) holds. In fact

$$v[F] = -2xy(x^2 + 1)^{-1}F(x, y)$$

Furthermore we can show that the maximal rank condition is satisfied since

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = (4x^3 + 2xy^2, 2x^2 y + 2y)$$

vanishes only at $(x, y) = (0, 0)$.

We can conclude that the solution set

$$\{(x, y) : x^4 + x^2 y^2 + y^2 = 1\}$$

is a rotationally invariant subset of $\mathbb{R}^2$.

Note that $F(x, y)$ is not a $G$-invariant function in this case. In fact most other level sets of $F$ are not rotationally invariant. See figure 1.6.

Therefore applying theorem 1.2.8 $SO(2)$ is a symmetry group of $F(x, y) = 0$

**Example 1.2.3.** Consider the one-parameter group generated by the vector field

$$v = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + (1 + z^2) \frac{\partial}{\partial z}$$
the group transformations are

$$\exp(\epsilon v)(x, y, z) = (x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon, \frac{\sin \epsilon + z \cos \epsilon}{\cos \epsilon - z \sin \epsilon})$$

The characteristic system is thus

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1 + z^2}$$

Solving this we can find that the functions

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad w = \frac{xz + y}{yz + x}$$

form a complete system of functionally independent invariants, whose common level sets describe the integral curves

### 1.2.2 Groups and Differential Equations

It should now be clear what we mean by symmetric group admitted by a given equation. We can now introduce the less trivial concept of symmetry group of a differential equation.

For the sake of simplicity we will avoid the case of a system of differential equation with \( q \) dependent variables treating the simpler case of one differential equation with \( N \) independent variables \( x = (x_1, \ldots, x^N) \) which we can view as local coordinates on the Euclidean space \( X \simeq \mathbb{R}^N \) and a single dependent variable \( u \) coordinates on \( U \simeq \mathbb{R} \). Anyway we will refer Olver [Olv93] for the general case of \( q \) dependent variables. The total space will be the Euclidean space \( E = X \times U \simeq \mathbb{R}^{N+1} \) coordinatized by the independent and dependent variables. The symmetries we will focus on are diffeomorphisms on the space of independent and dependent variables:

$$\tilde{(x, u)} = \exp(\epsilon v)(x, u) = (X_\epsilon(x, u), U_\epsilon(x, u))$$

for some smooth functions \( X_\epsilon \) and \( U_\epsilon \).

These are often referred to as **point transformations** since they act pointwise on the total space \( E \). However, it is convenient to specialize, on occasion, to more restrictive classes of transformations. For example, base transformations are only allowed to act on the independent variables, and so have the form

$$\tilde{(x, u)} = \exp(\epsilon v)(x, u) = (X_\epsilon(x), u)$$

In the case of connected groups, the action of the group can be recovered from that of its associated **infinitesimal generators**. A general vector field

$$v = \sum_{i=1}^{N} \xi^i(x, u) \partial_{x_i} + \phi(x, u) \partial_u$$

on the space of independent and dependent variables generates a flow \( \exp(\epsilon v) \), which is a **local one-parameter group of point transformations** on \( E \).
The concept of a symmetry group of a differential equation can be specified either in terms of its solutions or its frame.

The first definition treats a symmetry group of a differential equation as a group of transformations mapping every solution of the equation into its solution. This definition dealing with the totality of solutions of a given differential equation is natural, but does not give a practical method for finding all symmetries of an equation.

The second definition treats a symmetry group of a differential equation as a group of transformations whose prolongation, i.e. extension to the derivatives involved in a differential equation in question, leaves invariant the frame of the differential equation. This differential algebraic definition does not assume knowledge of solutions. Moreover, it is crucial for the infinitesimal formulation of the invariance of differential equations and provide a practical method for calculating symmetries by solving so-called determining equations.

**Definition 1.2.9.** The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property of mapping solutions to other solutions.

A Lie point symmetry is characterized by an infinitesimal transformation which leaves the given differential equation invariant under the transformation of all independent and dependent variables.

We have first of all to explain how a given transformation \( \exp(\epsilon v) \) in a Lie group \( G \) acts on a function \( u = f(x) \). A solution to a differential equation will be described by a smooth function \( u = f(x) \). The graph of the function,

\[
\Gamma_f = \{(x, f(x)) : x \in \Omega \} \subset X \times U
\]

where \( \Omega \subset X \) is the domain of definition of \( f \), determines a regular \( N \)-dimensional submanifold of the total space \( E = X \times U \).

The transform of \( \Gamma_f \) by \( \exp(\epsilon v) \) is

\[
\exp(\epsilon v)(\Gamma_f) = \{(\tilde{x}, \tilde{u}) = \exp(\epsilon v)(x, u) : (x, u) \in \Gamma_f \}.
\]

Notice how the set \( \exp(\epsilon v)(\Gamma_f) \) does not need to be necessarily a graph of another function \( \tilde{u} = \tilde{f}(\tilde{x}) \). However we can state that, since \( G \) acts smoothly and the identity leaves \( \Gamma_f \) unchanged, near the identity the transform \( \exp(\epsilon v)(\Gamma_f) = \Gamma_{\tilde{f}} \) is the graph of some function \( \tilde{u} = \tilde{f}(\tilde{x}) \). We will then write \( \tilde{f} = \exp(\epsilon v)(f) \) and call \( \tilde{f} \) the transform of \( f \) by \( \exp(\epsilon v) \).

In general if we are to recover the transformed function under the group action \( \tilde{f} = \exp(\epsilon v)(f) \) we will proceed as follows.

Let us suppose the transformation is given in coordinates by

\[
(\tilde{x}, \tilde{u}) = \Psi_\epsilon(x, u) = (X_\epsilon(x, u), U_\epsilon(x, u))
\]
for some smooth functions $X_\epsilon(x, u)$ and $U_\epsilon(x, u)$. The graph $\Gamma_f$ is therefore given by

$$
\begin{cases}
\tilde{x} = X_\epsilon(x, f(x)) = X_\epsilon(1 \times f)(x) \\
\tilde{u} = U_\epsilon(x, f(x)) = U_\epsilon(1 \times f)(x)
\end{cases}
$$

In order to find $\tilde{f}$ explicitly we have to get rid of $x$ in the previous system.

We know that for some $\epsilon$ sufficiently near the identity, the Jacobian is not singular and thus we can invert everything obtaining

$$
x = (X_\epsilon(1 \times f))^{-1}(\tilde{x})
$$

This leads us to

$$
\exp(\epsilon v)(f) = (U_\epsilon(1 \times f))(X_\epsilon(1 \times f))^{-1}
$$

This is far from being easy to compute. Let us notice anyway that in case the transformation acts on the independent variables alone the previous formula is way easier.

The transformation takes in fact the form

$$(\tilde{x}, \tilde{u}) = \exp(\epsilon v)(x, u) = (X_\epsilon(x, u), u)$$

Thus we can easily find

$$\tilde{u} = \tilde{f}(\tilde{x}) = f(X_\epsilon^{-1}(\tilde{x})) = f(X_{-\epsilon}(\tilde{x}))$$

**Example 1.2.4.** Consider the usual one-parameter group of rotations

$$
\exp(\epsilon v)(x, u) = (x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \sin \epsilon)
$$

acting on the space $E \simeq \mathbb{R}^2$ consisting of one independent and one dependent variable. Such a rotation transforms a function $u = f(x)$ by rotating its graph. The equation for the transformed function $\tilde{f} = \exp(\epsilon v)(f)$ is given in implicit form

$$
\tilde{x} = x \cos \epsilon - f(x) \sin \epsilon, \quad \tilde{u} = x \sin \epsilon + f(x) \cos \epsilon
$$

so that $\tilde{u} = \tilde{f}(\tilde{x})$ is found by eliminating $x$ from these two equations.

For example, if $u = ax + b$ is affine then the point $(x, u) = (x, ax + b)$ on the graph of $f$ is rotated to the point

$$(\tilde{x}, \tilde{u}) = (x \cos \epsilon - (ax + b) \sin \epsilon, x \sin \epsilon + (ax + b) \cos \epsilon)$$

Making $x$ explicit

$$
x = \frac{\tilde{x} + b \sin \epsilon}{\cos \epsilon - a \sin \epsilon}
$$

the transformed function is thus given by

$$
\tilde{u} = \frac{\sin \epsilon + a \cos \epsilon}{\cos \epsilon - a \sin \epsilon} + \frac{b}{\cos \epsilon - a \sin \epsilon}
$$

(1.31)
1.2 Lie Group of Transformations for PDE’s

which is defined provided \( \cot \epsilon \neq a \), i.e., provided the graph of \( f \) has not been rotated to be vertical.

**Definition 1.2.10.** A function \( u = f(x) \) is said to be invariant under the transformation group \( G \) if its graph \( \Gamma_f \) is a \( G \)-invariant subset.

**Example 1.2.5.** The graph of any invariant function \( f \) for the rotation group introduced in example 1.1.3 acting on \( \mathbb{R}^2 \) must be an arc of a circle centred at the origin, so \( u = \pm \sqrt{c^2 - x^2} \). Note that there are no globally defined invariant functions in this case.

In general, since any invariant function’s graph must, locally, be a union of orbits, the existence of invariant functions passing through a point \( z = (x, u) \in E \) requires that the orbit \( O \) through \( z \) be of dimension at most \( N \), the number of independent variables, and, furthermore, be transverse i.e. its tangent space contains no vertical tangent directions. Since the tangent space to the orbit

\[ T O | _z = g | _z \]

agrees with the space spanned by the infinitesimal generators of \( G \), the transversality condition requires that, at each point, \( g | _z \) contain no vertical tangent vectors.

**Example 1.2.6.** The infinitesimal generator of the rotation group is

\[ v = -u \partial_x + x \partial_u \]

is vertical at \( u = 0 \). Thus, the rotation group fails the transversality criterion on the \( x \)-axis and there are no smooth, rotationally invariant functions \( u = f(x) \) passing through such points.

As usual, the most convenient characterization of the invariant functions is based on an infinitesimal condition. Since the graph of a function is defined by the vanishing of its components \( u - f(x) \) we have the following infinitesimal invariance condition

\[ 0 = v[u - f(x)] = \phi(x, u) - \sum_{i=1}^{N} \xi_i(x, u) \frac{\partial f}{\partial x_i} \]

which must hold whenever \( u = f(x) \), for every infinitesimal generator \( v \in g \). These first order partial differential equations are known in the literature as the **invariant surface conditions** associated with the given transformation group.

Let us now give a more rigorous definition of a symmetry group of a differential equations.

**Definition 1.2.11.** Let \( F(x, u^{(m)}) \) be a differential equation. A symmetry group of the PDE \( F(x, u^{(m)}) \) is a local group of transformations \( G \) acting on an open subset \( M \) of the space of independent and dependent variables for the PDE with the property that whenever \( u = f(x) \) is a solution of \( F(x, u^{(m)}) \) and whenever \( \exp(\epsilon v)(f) \) is defined for \( \epsilon \in G \) then \( u = \exp(\epsilon v)(f(x)) \) is also a solution of the PDE.
1.2.3 Prolongation

We have now to properly define the concept of differential equations with a concrete geometric object. In order to be able to do this we have to “prolong” the basic space $X \times U$ representing the space of dependent and independent variables to a space that can represents all the derivatives as well. In fact since we are interested in studying the symmetries of differential equations, we need to know not only how the group transformations act on the independent and dependent variables, but also how they act on the derivatives of the dependent variables. In the recent years geometers have formalized this geometrical construction through the general definition of the jet space (or bundle) associated with the total space of independent and dependent variables. The jet space coordinates will represent the derivatives of the dependent variables.

We thus introduce the total space $X \times U^{(m)}$, with $U^{(m)} := U \times U_1 \times \cdots \times U_m$ and each $U_n$ the space of the derivatives of order $n$, whose coordinates represent the independent, the dependent and the derivatives of the dependent variables up to order $m$. The total space $X \times U^{(m)}$ is called the $m$-th jet space of the underlying space $X \times U$.

Given a smooth function $u = f(x)$, $f : X \to U$ there is an induced function $u^{(m)} = \text{pr}^{(m)} f(x)$ called the $m$-th prolongation of $f$ defined by the equation

$$u^{(m)} = D^m f(x)$$

Thus $\text{pr}^{(m)} f$ is a function from $X$ to the space $U^{(m)}$ and for each $x \in X$, $\text{pr}^{(m)} f(x)$ is a vector whose entries represent the values of $f$ and all of its derivatives up to order $m$ at the point $x$.

For instance in the case $p = 2$ with $u = f(x, y)$ we have

$$\text{pr}^{(2)} f(x, y) = (u, u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = \left( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2} \right)$$

Let us now suppose $G$ is a local group of transformations acting on an open subset $M \subset X \times U$. There is an induced local action of $G$ on the $m$-th jet space $M^{(m)}$ called the $m$-th prolongation of $G$ denoted $\text{pr}^{(m)} G$. This prolongation is defined so that it transforms the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. More rigorously let us suppose $(x_0, u_0^{(m)})$ is a given point in $M^{(m)}$. Chose any smooth function $u = f(x)$ defined in a neighbourhood of $x_0$ whose graph lies in $M$ and has the given derivatives at $x_0$

$$u_0^{(m)} = \text{pr}^{(m)} f(x_0)$$

If $\epsilon \in G$ the transformed function $\exp(\epsilon v) f$ is defined in a neighbourhood of

$$(\tilde{x}_0, \tilde{u}_0) = \exp(\epsilon v)(x_0, u_0)$$

We thus determine the action of the prolonged group transformation $\text{pr}^{(m)} \epsilon$ on the point $(x_0, u_0^{(m)})$ by evaluating the derivatives of the transformed function $\exp(\epsilon v) f$ at $\tilde{x}_0$. Explicitly

$$\text{pr}^{(m)} \exp(\epsilon v)(x_0, u_0^{(m)}) = (\tilde{x}_0, \tilde{u}_0^{(m)})$$
where
\[ \tilde{u}_0^{(m)} = \text{pr}^{(m)}(\exp(\epsilon v)f)(\tilde{x}_0). \]

Let now
\[ F(x, u^{(m)}) = 0 \]
be a differential equations as usual in $N$ independent variables and a single dependent variable. We are assuming $F(x, u^{(m)})$ to be a smooth map from the jet space $F : X \times U^{(m)} \to \mathbb{R}$.

The differential equation determines a subvariety of the total jet space
\[ S_F = \{(x, u^{(m)}) : F(x, u^{(m)}) = 0\} \]
Therefore a smooth solution of the given differential equations is a smooth function $u = f(x)$ such that
\[ F(x, \text{pr}^{(m)}f(x)) = 0 \]
whenever $x$ lies in the domain of $f$. This is equivalent to the requirement that the graph
\[ \Gamma_f^{(m)} = \{(x, \text{pr}^{(m)}f(x))\} \]
of the m-th prolongation of $f$ be entirely contained in the variety defined by the equation
\[ \Gamma_f^{(m)} \subset S_F \]

**Example 1.2.7.** Let $N = 1$ and $M = \mathbb{R} \times \mathbb{R}$ and consider the rotation group $SO(2)$ considered in example 1.2.4. Given a function $u = f(x)$ the first prolongation is
\[ \text{pr}^{(1)}f(x) = (f(x), f'(x)) \]

For a rotation in the one-parameter group the first prolongation $\text{pr}^{(1)} \exp(\epsilon v)$ will act on the space $(x, u, p)$, where, in accordance with classical notation, we use $p$ to represent the derivative coordinate $u_x$. Given a point $(x_0, u_0, p_0)$, we choose the linear polynomial
\[ u = f(x) = p_0(x - x_0) + u_0 \]
as representative, noting that $f(x_0) = u_0$, $f'(x_0) = p_0$. The transformed function is given by (1.31), so
\[ \tilde{f}(\tilde{x}) = \frac{\sin \epsilon + p_0 \cos \epsilon}{\cos \epsilon - p_0 \sin \epsilon} \tilde{x} + \frac{u_0 - p_0 x_0}{\cos \epsilon - p_0 \sin \epsilon} \]
so that
\[ \tilde{p}_0 = \tilde{f}'(\tilde{x}_0) = \frac{\sin \epsilon + p_0 \cos \epsilon}{\cos \epsilon - p_0 \sin \epsilon} \]
which is defined provided $p_0 \neq \cot \epsilon$. Therefore the first prolongation of the rotation group is
explicitly given by
\[ \text{pr}^{(1)} \exp(\epsilon v)(x, u, p) = \left( x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \cos \epsilon, \frac{\sin \epsilon + \p \cos \epsilon}{\cos \epsilon - \p \sin \epsilon} \right) \]
where for convenience reason we have dropped the 0 subscript.

**Definition 1.2.12.** A point transformation \( g : M \rightarrow M \) acting on the space \( M = X \times U \) of independent and dependent variables is called a symmetry of the system of partial differential equations \( F(x, u^{(m)}) = 0 \) if, whenever \( u = f(x) \) is a solution to the differential equation, and the transformed function \( \tilde{f} = \exp(\epsilon v)f \) is well defined, then \( \tilde{u} = \tilde{f}(\tilde{x}) \) is also a solution to the equation.

**Theorem 1.2.13.** Let \( M \) be an opened subset of \( X \times U \) and suppose \( F(x, u^{(m)}) = 0 \) is a differential equation defined over \( M \) with corresponding subvariety \( S_F \subset M^{(m)} \). Suppose \( G \) is a local group of transformation acting on \( M \) whose prolongation leaves \( S_F \) invariant i.e. whenever \( (x, u^{(m)}) \in S_F \) we have \( \text{pr}^{(m)} \exp(\epsilon v)((x, u^{(m)})) \in S_F \forall \epsilon \in G \) such that this is defined. Then \( G \) is a symmetry group of the differential equation in the sense of definition 1.2.11.

As done previously with the group transformation we can define the prolongation even for the corresponding infinitesimal generator.

**Definition 1.2.14.** Let \( M \subset X \times U \) be open and suppose \( v \) is a vector field on \( M \), with corresponding one-parameter group \( \exp(\epsilon v) \). The \( m \)-th prolongation of \( v \), denoted by \( \text{pr}^{(m)}v \), is defined to be the infinitesimal generator of the corresponding prolonged one parameter group \( \text{pr}^{(m)}[\exp(\epsilon v)] \). Therefore
\[ \text{pr}^{(m)}v \bigg|_{(x, u^{(m)})} = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \text{pr}^{(m)}[\exp(\epsilon v)](x, u^{(m)}) \quad (1.32) \]
for any \( (x, u^{(m)}) \in M^{(m)} \).

A vector field on \( M \) will in general take the form
\[ \text{pr}^{(m)}v = v^* = \sum_{i=1}^{N} \xi^i(x, u^{(m)}) \frac{\partial}{\partial x_i} + \sum_{|\alpha| \leq m} \phi^\alpha(x, u^{(m)}) \frac{\partial}{\partial u^\alpha} \quad (1.33) \]
where with \( \alpha \) we have denoted all the multi-index of orders \( 0 \leq |\alpha| \leq m \). In the case of \( v^* \) be a prolongation \( \text{pr}^{(m)}v \) of a vector field
\[ v = \sum_{i=1}^{N} \xi^i(x, u) \frac{\partial}{\partial x_i} + \phi(x, u) \frac{\partial}{\partial u} \]
the coefficients of \( v^* \) must agree with the corresponding coefficients of \( v \).

Of course the prolonged group action when restricted to just the zero-th order variables agrees with the ordinary group action on \( M \).

**Example 1.2.8.** Let us consider the rotation group \( SO(2) \) acting on \( X \times U \simeq \mathbb{R}^2 \). The
corresponding infinitesimal generator is

\[ v - u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} \]

with

\[ \exp(\epsilon v)(x, u) = (x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \cos \epsilon) \]

being \( \epsilon \) the rotation angle.

We have shown in example 1.2.14 that the first prolongation takes the form

\[ \text{pr}^{(1)}[\exp(\epsilon v)](x, u, u_x) = (x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \cos \epsilon, \frac{\sin \epsilon + u_x \cos \epsilon}{\cos \epsilon - u_x \sin \epsilon}) \]

The first prolongation of \( v \) is obtained by differentiating these expressions with respect to \( \epsilon \) and setting \( \epsilon = 0 \). Thus we can easily show that

\[ \text{pr}^{(1)} v = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} (1 + u_x^2) \frac{\partial}{\partial u_x} \]  

(1.34)

Note that the first two term in (1.34) agrees with those in \( v \).

The following is the main theorem, due to Lie, that characterize the symmetry group \( G \) of a given PDE.

**Theorem 1.2.15 (Lie).** Let us suppose

\[ F(x, u^{(m)}) = 0 \]  

(1.35)

is a differential equation. A group of transformation \( G \) is a symmetry group of a non degenerate differential equation of the form (1.35) iff

\[ \text{pr}^{(m)} v[F(x, u^{(m)})] = 0 \quad \text{whenever} \quad F(x, u^{(m)}) = 0 \]  

(1.36)

for every infinitesimal generator \( v \) of \( G \).

The proof follows immediately from theorem 1.2.13.

Here with non degenerate we mean a partial differential equation of maximal rank and locally solvable as in the following definitions

**Definition 1.2.16 (Maximal rank condition).** Let be given a differential equation

\[ F(x, u^{(m)}) = 0 \]

The system is said to be of maximal rank if the \( 1 \times (N + N^{(m)}) \) Jacobian matrix

\[ J_F(x, u^{(m)}) = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial u^{(m)}}, \ldots \right) \]
of $F$ with respect to the variables $(x, u^{(m)})$ is of rank 1 whenever

$$F(x, u^{(m)}) = 0$$

**Definition 1.2.17** (local solvability). Let be given a differential equation

$$F(x, u^{(m)}) = 0$$

For any point $(x_0, u_0^{(m)})$ such that

$$F(x_0, u_0^{(m)}) = 0$$

there exist a solution $u = f(x)$ with

$$u_0^{(m)} = \text{pr}^{(m)} f(x_0)$$

We have the following result due to Lie.

**Theorem 1.2.18** (Lie). Let us consider the PDE defined on a manifold $M$

$$F(x, u^{(m)}) = 0$$

The set of all infinitesimal symmetries from a Lie algebra of vector fields on $M$. Moreover if this Lie algebra is finite dimensional the symmetry group of the system is a local Lie group of transformations acting on $M$.

The conditions (1.36) are known as the determining equations of the symmetry group for the differential equation; note in particular that they are only required to hold on points $(x, u^{(m)}) \in S_F$ satisfying the equations. If we substitute the explicit formula (1.33) for the coefficients of the prolongation $\text{pr}^{(m)} v$, we find that the determining equations form a large, over-determined, linear system of partial differential equations for the coefficients $\xi^i$ and $\phi$ of $v$. In almost every example of interest, the determining equations are sufficiently elementary that they can be explicitly solved, and thereby one can determine the complete symmetry group of the equation.

**Example 1.2.9.** Let us consider again the rotation group $G = SO(2)$ acting on $X \times U = \mathbb{R}^2$. Let us consider the first order ordinary differential equation

$$F(x, u, u_x) = (u - x)u_x + u + x = 0$$

(1.37)

The Jacobian matrix is thus

$$J = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u_x}\right) = (1 - u_x, 1 + u_x, u - x)$$
which is of rank 1 everywhere. Applying the infinitesimal generator of \(pr^{(1)}SO(2)\) we get

\[
pr^{(1)}v[F] = -u \frac{\partial F}{\partial x} + x \frac{\partial F}{\partial u} + (1 + u^2_x) \frac{\partial F}{\partial u_x} = \\
= -u(1 - u_x) + x(1 + u_x) + (1 + u^2_x)(u - x) = \\
= u_x ((u - u_x)u_x + +u + x) = u_x F
\]

Thus \(pr^{(1)}v[F] = 0\) whenever \(F = 0\). Thus the condition (1.36) is satisfied. We conclude that the rotation group transforms solution of equation (1.37) to other solutions.

### 1.2.4 The Prolongation Formula

Theorem 1.2.15 is an important tool since it connects symmetry group of a differential equation with the infinitesimal criterion of invariance of the PDE under the prolonged infinitesimal generator of the group. We are therefore just left to find an explicit formula for the prolongation of a vector field.

Since the task is far from being simple we begin from some easier case in order to state the general case in the next section.

As stated before the inversion is not trivial but if we are in the case of a group action which acts solely on the independent variables the computation is actually a lot easier. Let us place ourselves for the moment in this case. We are therefore considering the vector field

\[
v = \sum_{i=1}^{N} \xi^i(x) \frac{\partial}{\partial x_i}
\]

on the space \(M \subset X \times U\). The group transformations \(\exp(\epsilon v) = \exp(\epsilon v)\) are of the form

\[
(\tilde{x}, \tilde{u}) = \exp(\epsilon v)(x, u) = (X_\epsilon(x), u)
\]

where \(\tilde{x}_i = X^i_\epsilon(x)\) has to satisfy

\[
\frac{dX^i_\epsilon(x)}{d\epsilon} \bigg|_{\epsilon=0} = \xi^i(x)
\]

We will denote now \(u_j := \frac{\partial u}{\partial x_j}\).

As stated in (1.30) we have

\[
\tilde{u} = \tilde{f}(\tilde{x}) = f (X^{-1}_\epsilon(\tilde{x})) = f (X^{-\epsilon}(\tilde{x}))
\]

Thus

\[
\tilde{u}_j = \frac{\partial \tilde{f}}{\partial \tilde{x}_j}(\tilde{x}) = \sum_{k=1}^{N} \frac{\partial f}{\partial x_k} (X^{-\epsilon}(\tilde{x})) \frac{\partial X^k_{-\epsilon}}{\partial \tilde{x}_j}(\tilde{x})
\]
We have that \( X_\epsilon (\tilde{x}) = x \) and therefore
\[
\tilde{u}_j = \sum_{k=1}^N \frac{\partial X^k_\epsilon}{\partial \tilde{x}_j} (X_\epsilon (x)) u_k
\]
is the explicit formula for the prolonged group action on the first order derivatives.

To find the infinitesimal generator of \( \text{pr}_1^{(1)} v \) we have to differentiate with respect to \( \epsilon \) and evaluate it at the point \( \epsilon = 0 \).

\[
\text{pr}_1^{(1)} v = \sum_{i=1}^N \xi^i (x) \frac{\partial}{\partial x_i} + \sum_{j=1}^N \phi^j (x, u^{(1)}) \frac{\partial}{\partial u_j}
\]
and from (1.32)
\[
\phi^j (x, u^{(1)}) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \sum_{k=1}^N \frac{\partial X^k_\epsilon}{\partial \tilde{x}_j} (X_\epsilon (x)) u_k
\]
As standard assumption we required all the functions to be smooth such that we can interchange the order of differentiation obtaining
\[
\frac{\partial}{\partial \tilde{x}_j} \left[ \left. \frac{dX^k_\epsilon}{d\epsilon} \right|_{\epsilon=0} (X_\epsilon (x)) \right] = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \frac{\partial X^k_\epsilon}{\partial \tilde{x}_j} (X_\epsilon (x))
\]
where we have used (1.39) together with the fact that at \( \epsilon = 0 \) and \( X_0 (x) = x \) is the identity.

Furthermore we get
\[
\sum_l \frac{\partial^2 X^k_\epsilon}{\partial \tilde{x}_j \partial \tilde{x}_l} (X_\epsilon (x)) \left. \left[ \frac{dX^l_\epsilon}{d\epsilon} \right|_{\epsilon=0} \right] = 0
\]
It vanishes since, as already stated before, \( X_0 (x) = x \) is the identity. Thus at \( \epsilon = 0 \) all second order \( x \)-derivatives of \( X_\epsilon \) vanish.

We can now conclude that the basic prolongation formula for \( \text{pr}_1^{(1)} v \) we get
\[
\phi^j (x, u^{(1)}) = - \sum_{k=1}^N \frac{\partial \xi^k}{\partial \tilde{x}_j} u_k
\]
As briefly mentioned previously the coefficients \( \xi^i (x) \) has to coincide with the coefficients of \( v \).

We now introduce the last definition before finally stating the main theorem that will allows us to construct the symmetric group.

Before giving the formal definition of total derivative we have to explain why we cannot use the "classical" definition of derivative.

Let us consider a function \( F \) depending on a set of independent variables \( x = (x_1, \ldots, x_N) \) and a dependent variable \( u \) together with the set of all derivatives up to order \( m \) w.r.t. the independent variable \( x \) \( (u^{(1)}, \ldots, u^{(m)}) \). We are interested in the derivatives of these functions with respect to all independent variables. If we assume thus that \( u \) depends on \( x \), we must consider all derivatives of \( F \) with respect to \( x \) and \( u^{(m)} \). Basically we are treating \( u \) and its derivatives as a function of \( x \).

**Definition 1.2.19 (Total derivative)**. Let \( F (x, u^{(m)}) \) be a smooth function of \( x, u \) and derivatives of \( u \) up to order \( m \), defined on an open subset \( M^{(m)} \subset X \times U^{(m)} \). From now on we will assume that


$m$ is the maximal order of the derivatives.

The total derivative of $F$ with respect to $x^i$ is the unique smooth function $D_i F(x, u^{m+1})$ defined on $M^{(m+1)}$ and depending on derivatives of $u$ up to order $m + 1$ with the property that if $u = f(x)$ is any smooth function

$$D_i F(x, pt^{(m+1)} f(x)) = \frac{\partial}{\partial x^i} \{ F(x, pt^{(m)} f(x)) \}$$

Therefore $D_i F$ is obtained from $P$ by differentiating $F$ with respect to $x^i$ while treating all the $u$'s and its derivatives as functions of $x$.

**Proposition 1.2.20.** Given $F(x, u^{(m)})$, the $i$-th total derivative of $F$ has the general form

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha} u^\alpha_i \frac{\partial F}{\partial u^\alpha}$$

where for $\alpha = (\alpha_1, \ldots, \alpha_k)$

$$u^\alpha_i \equiv \frac{\partial u^\alpha}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_k}}$$

In (1.40) the sum is over all the multi-indices of order $|\alpha| \leq m$ where $m$ is the highest order derivative appearing in $P$.

### 1.2.5 The General Prolongation Formula

We have finally all the necessary tools to introduce the final and key theorem.

**Theorem 1.2.21.** Let

$$v = \sum_{i=1}^N \xi_i(x, u) \frac{\partial}{\partial x^i} + \phi(x, u) \frac{\partial}{\partial u}$$

be a vector field defined on an open subset $M \subset X \times U$. The $m$-th prolongation of $v$ is the vector field

$$pr^{(m)} v = v + \sum_{\alpha} \phi^\alpha(x, u^{(m)}) \frac{\partial}{\partial u^\alpha}$$

defined on the corresponding space $M^{(m)} \subset X \times U^{(m)}$, the second summation being over the multi-index $\alpha$ of order $0 \leq |\alpha| \leq m$.

The coefficient functions $\phi^\alpha$ of $pr^{(m)} v$ are given by

$$\phi^\alpha(x, u^{(m)}) = D_\alpha \left( \phi - \sum_{i=1}^N \xi_i u_i \right) + \sum_{i=1}^N \xi_i^\alpha u_i$$

where we have denoted by $u_i = \frac{\partial u}{\partial x^i}$ and $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$.

**Proof.** We start by proving the case $n = 1$.

Let $\exp(\epsilon v) = \exp(\epsilon \xi)$ the corresponding one-parameter group with transformations

$$(\tilde{x}, \tilde{u}) = \exp(\epsilon v)(x, u) = (X_\epsilon(x, u), U_\epsilon(x, u))$$
Notice that
\[
\begin{align*}
\xi_i(x,u) &= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} X_i(x,u), \quad i = 1, \ldots, p \\
\phi(x,u) &= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} U_i(x,u)
\end{align*}
\]
(1.43)

Let \( u = f(x) \) and \( u^{(1)} = \text{pr}^{(1)} f(x) \). Then for \( \epsilon \) sufficiently small the transform of \( f \) by the group element \( \exp(\epsilon v) \) is well-defined and \( x \) is given by
\[
\ddot{u} = \hat{f}(\hat{x}) = \exp(\epsilon v)(f, \hat{x}) = (U_\epsilon \cdot (1 \times f)) \cdot (X_\epsilon \cdot (1 \times f))^{-1}(\hat{x})
\]

From the chain rule we obtain
\[
J\hat{f}(\hat{x}) = J(U_\epsilon \cdot (1 \times f))(x) \cdot (J(X_\epsilon \cdot (1 \times f))(x))^{-1}
\]
(1.44)
where \( J \) denote here the Jacobian matrix and \( x = (X_\epsilon \cdot (1 \times f))(x))^{-1}(\hat{x}) \). Writing out the matrix entries of \( J\hat{f}(\hat{x}) \) thus provides explicit formulae for the first prolongation \( \text{pr}^{(1)} v \).

To find the infinitesimal generator \( \text{pr}^{(1)} v \) we have to differentiate (1.44) with respect to \( \epsilon \) and evaluating it at \( \epsilon = 0 \). Recall first given an invertible matrix \( M(\epsilon) \)
\[
\frac{d}{d\epsilon} (M(\epsilon))^{-1} = -M(\epsilon)^{-1} \frac{dM(\epsilon)}{d\epsilon} M(\epsilon)^{-1}
\]

Notice furthermore that since \( \epsilon = 0 \) corresponds to the identity transformation
\[
X_0(x, f(x)) = x, \quad U_0(x, f(x)) = f(x)
\]
(1.45)
so denoting by \( I \) the \( p \times p \) identity matrix
\[
J(X_0 \cdot (1 \times f))(x) = I, \quad J(U_0 \cdot (1 \times f))(x) = Jf(x)
\]

Differentiating now (1.44) and setting \( \epsilon = 0 \) we find
\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} J\hat{f}(\hat{x}) = J(U_\epsilon \cdot (1 \times f))(x) \cdot (J(X_\epsilon \cdot (1 \times f))(x))^{-1}
\]
\[
= J(\phi \cdot (1 \times f))(x) - Jf(x)J(\xi \cdot (1 \times f))(x)
\]

Here \( \xi \) and \( \phi \) are evaluated as (1.43).

Therefore the \((k) - th\) entry is
\[
\phi^k_\alpha(x, \text{pr}^{(1)} f(x)) = \frac{\partial}{\partial x^k} (\phi_\alpha(x, f(x))) - \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_k} (\xi^i(x, f(x)))
\]
and from the definition of total derivative
\[
\phi^k(x, u^{(1)}) = D_k(\phi(x, u) - \sum_{i=1}^{N} D_k(\xi^i(x, u))u_i = \quad (1.46)
\]
\[ D_k \left( \phi - \sum_{i=1}^{N} \xi^i u_i \right) + \sum_{i=1}^{N} \xi^i u_{ki} \]

This conclusion the proof for \( n = 1 \).

The general case is carried out by induction. The key is to see \( M^{(n+1)} \) as a subspace of \( (M^{(n)})^{(1)} \).

Therefore the inductive procedure for determining \( pr^{(m-1)}v \) as a vector field on \( M^{(m-1)} \) and by first order prolongation formula can be prolonged to \( (M^{(m-1)})^{(1)} \). We then restrict the resulting vector field to the subspace \( M^{(m)} \) and this will determine the \( m-th \) prolongation \( pr^{(m)}v \). According to (1.46) the coefficient of \( \frac{\partial}{\partial u_i^k} \) in the first prolongation of \( pr^{(m-1)}v \) is therefore

\[ \phi^k = 1 D_k \phi^\alpha - \sum_{i=1}^{N} D_k \xi^i u_i^\alpha \]  

(1.47)

It is now sufficient to check that (1.42) solves (1.47). By induction we find

\[ \phi^k = D_k \left\{ D\alpha \left( \phi - \sum_{i=1}^{N} \xi^i u_i \right) + \sum_{i=1}^{N} \xi^i u_i^\alpha \right\} - \sum_{i=1}^{N} D_k \xi^i u_i^\alpha = \]

\[ D_k D\alpha \left( \phi - \sum_{i=1}^{N} \xi^i u_i \right) + \sum_{i=1}^{N} (D_k \xi^i u_i^\alpha + \xi^i u_k^\alpha) - \sum_{i=1}^{N} D_k \xi^i u_i^\alpha \]

\[ = D_k D\alpha \left( \phi - \sum_{i=1}^{N} \xi^i u_i \right) + \sum_{i=1}^{N} \xi^i u_k^\alpha \]

Thus \( \phi^k \) is of the form (1.42) and the theorem is proved.

Given therefore an infinitesimal generator of the form

\[ v = \sum_{i=1}^{P} \xi_i(x, u) \frac{\partial}{\partial x_i} + \phi(x, u) \frac{\partial}{\partial u} \]

we have to solve the system of ODE

\[ \begin{aligned} 
\frac{d\tilde{x}_i}{de} &= \xi_i(\tilde{x}, \tilde{u}), \quad \tilde{x}_i(0) = x_i \\
\frac{d\tilde{u}}{de} &= \phi(\tilde{x}, \tilde{u}), \quad \tilde{u}(0) = u 
\end{aligned} \]  

(1.48)

Then the new solution will be \( \tilde{u}(\tilde{x}) \). As already said we will refer to this procedure as exponentiating the vector field \( v \).

All the theorems stated in the present section lead to the formulation of Lie’s algorithm for finding symmetries of an m-order PDE \( F(x, u^{(m)}) \) which is assumed to be solvable and of maximal rank. In particular the algorithm proceed as follows:

1. given an m-order PDE \( F(x, u^{(m)}) \) calculate \( pr^{(m)}v[F(x, u^{(m)})] \);

2. substitute the PDE in the expression of step 1 eliminating redundant information setting
3. extract the determining equations from the prolongation by setting the coefficient of the derivatives in the dependent variables equal to 0;

4. solve the resulting determining equations.

The previous procedure will be clearer in a while when the complete method will be applied to the Heat equation.

Example 1.2.10. A general vector field on $X \times U \simeq \mathbb{R}^2 \times \mathbb{R}$ takes the form

$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u}$$

The first prolongation of $v$ is

$$\text{pr}^{(1)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t}$$

where

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} \tau u_{xt}$$

$$= D_x \phi - u_x D_x \xi - u_t D_x \tau$$

$$= \phi_x + (\phi_u - \xi_x) u_x - \tau_x u_t - \xi u_x^2 - \tau_u u_x u_t$$

(1.49)

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} \tau u_{tt}$$

$$= D_t \phi - u_x D_t \xi - u_t D_t \tau$$

$$= \phi_t + (\phi_u - \tau_t) u_t - \xi u_t - \tau_u u_t^2 - \xi u_x u_t$$

(1.50)

Similarly

$$\text{pr}^{(2)}v = \text{pr}^{(1)}v + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{xt} \frac{\partial}{\partial u_{xt}}$$

Example 1.2.11. Let us consider again the rotation group $SO(2)$ with infinitesimal generator

$$v = -u \partial_x + x \partial_u$$

The first prolongation is thus

$$\text{pr}^{(1)}v = v + \phi^x \partial_{u_x}$$

where

$$\phi^x = D_x(\phi - \xi u_x) + \xi u_{xx} = D_x(x + xu_x) - uu_{xx} = 1 + u_x^2$$

as already evaluated previously in example 1.2.9.
1.3 Invariance of a PDE

Using formula (1.42) we further get

\[ \phi^{xx} = D^2_x(\phi - \xi u_x) + \xi u_{xxx} = D^2_x(x + uu_x) - uu_{xxx} = 3u_x u_{xx} \]

Thus the infinitesimal generator of the second prolongation acting on \( X \times U^{(2)} \) is

\[ pr^{(2)}v = -u \partial_x + x \partial_u + (1 + u^2_x) \partial_u + 3uu_{xx} \partial_{u_{xx}} \]

Using now theorem (1.2.15) we can deduce that the ordinary differential equation

\[ u_{xx} = 0 \]

has \( G = SO(2) \) has a symmetry group since

\[ pr^{(2)}v[u_{xx}] = 3u_x u_{xx} = 0 \quad \text{whenever} \quad u_{xx} = 0 \]

This is obvious since a rotation clearly takes straight lines to straight lines.

Let us consider further the function

\[ F(x, u^{(2)}) = u_{xx}(1 + u^2_x)^{-\frac{3}{2}} \]

We can easily show that

\[ pr^{(2)}v[F] = 0 \]

for all \( u_x \) and \( u_{xx} \). Hence by proposition 1.2.5, \( F \) is an invariant of \( pr^{(2)}SO(2) \)

\[ F(pr^{(2)} \exp(\epsilon v)(x, u^{(2)})) = F(x, u^{(2)}) \]

for any rotation \( \epsilon \). But \( F \) is just the curvature of curve determined by the graph of \( u = f(x) \), so we have just reproved the fact that the curvature of a curve is invariant under rotations.

1.3 Invariance of a PDE

We have now seen how to determine internal symmetries admitted by a given differential equation in a systematic way. In this section we will apply infinitesimal transformations to the construction of solutions of partial differential equations.

Invariants surface of the corresponding Lie group of point transformations leads to invariant solutions. These solutions are obtained by solving a differential equation with fewer independent variables than the original one.

If a one-parameter Lie group of transformations admitted by the PDE leaves the domain and the boundary condition of of a BVP invariant, then the solution of the BVP is an invariant solution and hence the given BVP is reduced to a BVP with one less independent variable.
1.3.1 Recall on the notion of invariants

We will now just recall (and actually give a better characterization) of notions of invariant defined in previous sections.

Let us now consider a one-parameter Lie group of transformations

\[ \tilde{x} = \exp(\epsilon \mathbf{v})(x) \]  

(1.51)

with infinitesimal generator

\[ \mathbf{v} = \sum_{i=1}^{N} \xi^i(x, u) \frac{\partial}{\partial x_i} \]  

(1.52)

**Definition 1.3.1 (Invariant surface).** A surface \( F(x) = 0 \) is an invariant surface for a one-parameter Lie group of transformations (1.51) with infinitesimal generator (1.52) iff

\[ F(\tilde{x}) = 0 \text{ when } F(x) = 0 \]

**Definition 1.3.2 (Invariant curve).** A curve \( F(x, y) = 0 \) is an invariant curve for a one-parameter Lie group of transformations

\[
\begin{cases}
\tilde{x} = X(x, y) \\
\tilde{y} = Y(x, y)
\end{cases}
\]  

(1.53)

with infinitesimal generator

\[ \mathbf{v} = \xi^x(x, y) \frac{\partial}{\partial x} + \xi^y(x, y) \frac{\partial}{\partial y} \]

iff

\[ F(\tilde{x}, \tilde{y}) = 0 \text{ when } F(x, y) = 0 \]

**Definition 1.3.3 (Invariant point).** A point \( x \) is an invariant point for a one-parameter Lie group of transformations (1.51) with infinitesimal generator (1.52) iff \( \tilde{x} \equiv x \) under (1.51).

**Theorem 1.3.4.** (i) A surface written in a solved form \( F(x) = x_n - f(x_1, \ldots, x_{n-1}) = 0 \) is an invariant surface for (1.51) iff

\[ \mathbf{v}[F(x)] = 0 \text{ when } F(x) = 0. \]

(ii) A curve written in a solved form \( F(x, y) = y - f(x) = 0 \) is an invariant curve for (1.53) iff

\[ \mathbf{v}[F(x, y)] = \xi^y(x, y) - \xi^x(x, y)f'(x) = 0 \]

when

\[ F(x, y) = y - f(x) = 0 \]

i.e. iff

\[ \xi^y(x, f(x)) - \xi^x(x, f(x))f'(x) = 0 \]
Theorem 1.3.4 gives a way of finding invariant surface of a given Lie group of transformations. We will now see an application of the previous theorem.

**Definition 1.3.5.** The family of surfaces

\[ \omega(x) = \text{const} = c \]

is an invariant family of surfaces for (1.51) iff

\[ \omega(\tilde{x}) = \text{const} = \tilde{c} \quad \text{when} \quad \omega(x) = c \]

**Definition 1.3.6.** The family of curves

\[ \omega(x, y) = \text{const} = c \]

is an invariant family of curves for (1.53) iff

\[ \omega(\tilde{x}, \tilde{y}) = \text{const} = \tilde{c} \quad \text{when} \quad \omega(x, y) = c \]

From these definitions it immediately follows that

\[ \tilde{c} = \exp(\epsilon v^c(c)) \]

for some function \( \exp(\epsilon v^c) \) of \( c \) and group parameter \( \epsilon \).

**Theorem 1.3.7.** (i) A family of surfaces surface \( \omega(x) = \text{const} = c \) is an invariant family of surfaces for (1.51) iff

\[ v[\omega] = F_\omega(\omega) \]

for some infinitely differentiable function \( F_\omega(\omega) \).

(ii) A family of curves \( \omega(x, y) = \text{const} = c \) is an invariant family of curves for (1.53) iff

\[ v[\omega] = \xi^x(x, y) \frac{\partial \omega}{\partial x} + \xi^y(x, y) \frac{\partial \omega}{\partial y} = F_\omega(\omega) \]

for some infinitely differentiable function \( F_\omega(\omega) \).

In order to find the invariant family of surfaces for a Lie group of transformations we can set, without loss of generality, \( F_\omega(\omega) \equiv 1 \). This follows from the fact that if \( \omega(x) = c \) is an invariant family of surfaces then so is \( F(\omega(x)) = F(c) \) for any function \( F \). Further \( v[F(\omega(x))] = F'(\omega) v[\omega] = F'(\omega) F_\omega(\omega) \), so that setting \( F'(\omega) = \frac{1}{F_\omega(\omega)} \), we have \( v[F(\omega)] \equiv 1 \).

We are now considering the PDE

\[ F(x, u^{(m)}) = 0 \quad (1.54) \]

The notation is as before.
Definition 1.3.8. The one-parameter Lie group of transformation

\[
\begin{aligned}
\tilde{x} &= X_\epsilon(x, u) \\
\tilde{u} &= U_\epsilon(x, u)
\end{aligned}
\] (1.55)

leaves (1.54) invariant if and only if its m-th prolongation leaves the surface (1.54) invariant.

Invariance of the surface (1.54) under the m-th extension of (1.55) means that any solution \( u = f(x) \) is mapped into some other solution \( u = \phi(x; \epsilon) \) of (1.54) under the action of the group (1.55).

Consequently the family of all solutions of PDE (1.54) is invariant under (1.55) iff (1.54) admits (1.55).

We recall the fundamental result on determining equations stated in theorems 1.2.15 and 1.2.18.

Theorem 1.3.9 (Infinitesimal criterion for invariance of a PDE). Let

\[
v = \sum_{i=1}^{N} \xi^i(x, u) \frac{\partial}{\partial x_i} + \phi(x, u) \frac{\partial}{\partial u}
\] (1.56)

be the infinitesimal generator of (1.55). Let

\[
pr^{(m)}[v] = \sum_{i=1}^{N} \xi^i(x, u^{(m)}) \frac{\partial}{\partial x_i} + \sum_{\alpha} \phi^\alpha(x, u^{(m)}) \frac{\partial}{\partial u^\alpha}
\]

the m-th prolongation of the infinitesimal generator of (1.56).

Then (1.55) is admitted by the PDE (1.54) iff

\[
pr^{(m)}[v] \left[ F(x, u^{(m)}) \right] = 0
\]

when

\[
F(x, u^{(m)}) = 0
\]

Example 1.3.1. Let us consider the second-order PDE

\[
u_{xx} = g(x, u, u_x, u_y, u_{xy}, u_{yy})
\] (1.57)

The PDE (1.57) admits the one-parameter Lie group of translation

\[
\begin{aligned}
\tilde{x} &= X_\epsilon(x) = x \\
\tilde{y} &= Y_\epsilon(y) = y + \epsilon \\
\tilde{u} &= U_\epsilon(u) = u
\end{aligned}
\] (1.58)

since under this transformation we have

\[
\tilde{u}_{xx} = u_{xx}, \quad \tilde{u}_{xy} = u_{xy}, \quad \tilde{u}_{yy} = u_{yy}, \quad \tilde{u}_x = u_x, \quad \tilde{u}_y = u_y
\]
such that the surface defined by (1.57) is invariant in the \((x, u, u^{(1)}, u^{(2)})\)-jet space. Then for any function \(f(x)\) we have

\[ u = f(x) \]

is invariant under the transformation (1.58) and defines an invariant solution of (1.57) provided that \(f(x)\) solves the second order PDE

\[ f''(x) = g(x, f(x), f'(x), 0, 0, 0) \]

Note now that

\[ u = f(x, y) \]

defines an invariant surface of the transformation (1.58) iff

\[ v(u - f(x, y)) = -\frac{\partial f}{\partial y} = 0 \quad \text{when} \quad u = f(x, y) \]

where \(v = \frac{\partial}{\partial y}\) is the infinitesimal generator of (1.58)

**Example 1.3.2.** Let us now consider the wave equation

\[ u_{tt} = u_{xx} \tag{1.59} \]

Equation (1.59) admits the one-parameter group of scaling

\[
\begin{align*}
\tilde{t} = T_\epsilon(t) &= \epsilon t \\
\tilde{x} = X_\epsilon(x) &= \epsilon x \\
\tilde{u} = U_\epsilon(u) &= u 
\end{align*}
\]

(1.60)
since

\[ \tilde{u}_{tt} = \frac{1}{\epsilon} u_{tt}, \quad \tilde{u}_{xx} = \frac{1}{\epsilon} u_{xx} \]

and thus

\[ \tilde{u}_{tt} = \tilde{u}_{xx} \quad \text{when} \quad u_{tt} = u_{xx} \]

If we change the coordinate choosing

\[ r = \frac{t}{x}, \quad s = \ln(x), \quad u \]

thus (1.60) becomes

\[ \tilde{r} = r \hat{s} = s + \ln(\epsilon) \tilde{u} = u \]

then the wave equation turn into the PDE

\[ (1 - r^2) u_{rr} - u_{ss} + 2ru_{rs} + u_s - 2ru_r = 0 \tag{1.61} \]
Correspondingly
\[ u = f(r) = f\left(\frac{t}{x}\right) \] (1.62)
defines an invariant solution of (1.61) and hence of the wave equation (1.59) provided that \( f(r) \) satisfies the ODE
\[ (1 - r^2)f''(r) - 2rf'(r) = 0 \]
The infinitesimal generator of the transformation is given by
\[ v = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \]
Therefore \( u = f(t, x) \) defines an invariant surface of the transformation (1.60) iff
\[ v[u - f(t, x)] = 0 \quad \text{when} \quad u = f(t, x) \]
iff
\[ t \frac{\partial f}{\partial t} + x \frac{\partial f}{\partial x} = 0 \]
The corresponding characteristic equations
\[ \frac{dt}{t} = \frac{dx}{x} = \frac{df}{0} \]
lead to the invariant form (1.62).

### 1.3.2 Invariance solutions and similarity reduction

We have since now just recalled notions already introduced previously. We are now to discuss the invariance condition of a partial differential equation to reduce the equation to an ODE or to a PDE with less independent variables. Such procedure has many idea in common (it is actually a "generalization") with a branch of analysis know as dimensional analysis. For a mild treatment of the topic we refer to the first chapter of Bluman and Kumei [BK89].

The reduction procedure generates a similarity representation (a physical interpretation of a similarity solution is a flow which "looks the same" either at all times, or at all length scales) of the original equation. Thus a similarity reduction results into a representation which has some advantage compared with the original equation.

Let us consider again a PDE of the from (1.54) wich admits a one-parameter Lie group of transformations with infinitesimal generator (1.56).

**Definition 1.3.10.** A solution \( u = f(x) \) is an invariant solution of (1.54) corresponding to (1.56) admitted by (1.54) iff

(i) \( u = f(x) \) is an invariant surface of (1.56);

(ii) \( u = f(x) \) solves (1.54).

Therefore we can say that \( u = f(x) \) is an invariant solution of (1.54) iff \( u \) satisfies
1.3 Invariance of a PDE

(i) \( v[u - f(x)] = 0 \) when \( u = f(x) \) i.e

\[
\xi^i(x,u) \frac{\partial u}{\partial x_i} = \phi(x,u);
\]  \hspace{1cm} (1.63)

(ii) \( F(x, u^{(m)}) = 0 \).

Equation (1.63) is called invariant surface condition for invariant solutions.

Invariants solutions can be determined in two ways:

Invariant Form Method

**Theorem 1.3.11.** A one-parameter local group \( G \) of transformations in \( \mathbb{R}^N \) has precisely \( N - 1 \) functionally independent invariants. Any set of independent invariants

\[
(\psi_1(x) = C_1, \ldots, \psi_{N-1}(x) = C_{N-1})
\]
determines a basis of invariants of \( G \). The basis of course is not unique.

An arbitrary invariant \( F(x) \) of \( G \) is given by the formula

\[
F = \Phi(\psi_1(x), \ldots, \psi_{N-1}(x))
\]

With this method we solve the first order PDE (1.63) by solving the corresponding characteristic equation for \( u = f(x) \)

\[
\frac{dx_1}{d\xi^1(x,u)} = \frac{dx_2}{d\xi^2(x,u)} = \cdots = \frac{dx_N}{d\xi^N(x,u)} = \frac{du}{d\phi(x,u)}
\]  \hspace{1cm} (1.64)

If

\[
(\psi_1(x,u), \ldots, \psi_{N-1}(x,u)), F(x,u)
\]
are \( N \) independents invariants of (1.64) then the solution \( u = f(x) \) is given implicitly by the invariant form

\[
F(x,u) = \Phi(\psi_1(x,u), \ldots, \psi_{N-1}(x,u))
\]  \hspace{1cm} (1.65)

where \( \Phi \) is an arbitrary function of \( (\psi_1(x,u), \ldots, \psi_{N-1}(x,u)) \).

This reduced order PDE is obtained by substituting (1.65) into (1.54). It is straightforward that if \( N = 2 \) then the reduced PDE is an ODE.

**Theorem 1.3.12.** Let us now consider the change of coordinates \( (x'_1, \ldots, x'_N) \) for the group \( G \) of transformations defined by

\[
x'_1 = \phi(x), \quad x'_2 = \psi_1(x) \quad \ldots \quad x'_N = \psi_{N-1}(x)
\]

where \( \phi \) is a solution of

\[
v[\phi] = 1
\]
and \((\psi_1(x), \ldots, \psi_N(x))\) is a basis of invariants of the group \(G\). Then \((x'_1, \ldots, x'_N)\) is a set of canonical coordinates.

Recall that for canonical variables exist for any one-parameter group. It is a change of coordinates such that under the new coordinate system the transformation is the translation group.

**Example 1.3.3.** Let us consider the rotation group \(G = SO(2)\) defined by

\[
\begin{align*}
\tilde{x} &= x \cos \epsilon - y \sin \epsilon \\
\tilde{y} &= x \sin \epsilon + y \cos \epsilon
\end{align*}
\]  

with infinitesimal generator

\(v = -y \partial_x + x \partial_y\)

Solving the system

\[
\frac{dx}{-y} = \frac{dy}{x}
\]

leads to the invariant

\(\rho = \sqrt{x^2 + y^2}\)

with

\(v[\rho] = 2xy - 2xy = 0\)

We now know that any other invariants of the rotation group is of the form \(\theta(y) = \Phi(\rho)\). From theorem 1.3.12 we require

\(v[\theta] = 1\)

Thus we get

\[
x \frac{d\theta}{dy} = 1 \implies \frac{d\theta}{dy} = \frac{1}{\sqrt{\rho^2 - y^2}} \implies \theta = \arctan \left( \frac{y}{\sqrt{\rho^2 - y^2}} \right)
\]

Recalling the formula

\(\arctan(x) = \arcsin \left( \frac{x}{\sqrt{1 + x^2}} \right)\)

we get

\(\theta = \arcsin \left( \frac{y}{\rho} \right)\)

Thus the canonical coordinates for the rotation group are the polar coordinates

\((\rho, \theta) = (\sqrt{x^2 + y^2}, \arcsin \left( \frac{y}{\rho} \right))\)
Explication now fo \( (x(\theta, \rho), y(\theta, \rho)) \) leads to

\[
\begin{align*}
  x &= \rho \cos \theta \\
  y &= \rho \sin \theta
\end{align*}
\]  

(1.67)

and we can easily seen that the transformation is expressed as a translation in the form

\[
\begin{align*}
  \hat{\rho} &= \rho \\
  \hat{\theta} &= \theta + \epsilon
\end{align*}
\]  

(1.68)

with infinitesimal generator

\[ v = \partial_\theta \]

**Direct Substitution Method** This method in used when it is impossible to solve the invariant surface condition (1.63).

Without loss of generality we assume \( \xi^a(x, u) \neq 0 \). Then (1.63) becomes

\[
 u_n = -\sum_{i=1}^{n-1} \frac{\xi^i(x, u)}{\xi^n(x, u)} u_i + \frac{\phi(x, u)}{\xi^n(x, u)}
\]  

(1.69)

Therefore after substituting (1.69) into (1.54) for all term involving derivatives of \( u \) with respect to \( x^a \) we obtain a reduced order PDE with dependent variable \( u \) and independent variables \( (x^1, \ldots, x^{n-1}) \) and parameter \( x^n \).

### 1.3.3 Formulation of invariace of a BVP for a PDE

The application of infinitesimal transformations to BVP’s for PDE’s is much more restrictive than for the ODE’s. In fact if for an ODE any BVP is automatically reduced to a BVP for the reduced order ODE, it not usually hold for PDE’s as well. In the case of a PDE an invariant solution arising from an admitted infinitesimal generator solves a given BVP if the infinitesimal generator leaves all boundary conditions invariant. If the PDE is linear anyway the situation is a little bit less restrictive.

Let us as usual consider a PDE of the form

\[ F(x, u^{(m)}) = 0 \]  

(1.70)

defined on a domain \( \Omega \) with boundary conditions

\[ B_J(x, u^{(m-1)}) = 0 \]  

(1.71)

prescribed on the boundary surfaces

\[ \omega_J = 0, \quad J = 1, \ldots, s \]  

(1.72)
Let us further assume that the BVP has a unique solution. Consider now an infinitesimal generator of the form
\[
v = \sum_{i=1}^{N} \xi^i(x,u) \frac{\partial}{\partial x_i} + \phi(x,u) \frac{\partial}{\partial u}
\] (1.73)
which defines a one-parameter Lie group of transformations in x-space as well in (x,u)-space.

**Definition 1.3.13.** The infinitesimal generator \(v\) is admitted by the BVP iff

(i) 
\[\text{pr}^{(m)} v \left[ F(x,u^{(m)}) \right] = 0 \text{ when } F(x,u^{(m)}) = 0; \] (1.74)

(ii) 
\[v[\omega_J(x)] = 0 \text{ when } \omega_J(x) = 0, \quad J = 1, \ldots, s;\] (1.75)

(iii) 
\[\text{pr}^{(m-1)} v \left[ B_J(x,u^{(m-1)}) \right] = 0 \text{ when } B_J(x,u^{(m-1)}) = 0 \text{ on } \omega_J(x) = 0, \quad J = 1, \ldots, s\] (1.76)

**Theorem 1.3.14.** Let the BVP admits the infinitesimal generator (1.73). Let
\[(\psi_1(x,u), \ldots, \psi_{N-1}(x,u))\]
be N-1 independent group invariants of (1.73) depending only on x. Let \(F(x,u)\) be a group invariant of (1.73). Then the BVP reduces to
\[G(X,F^{(m)}) = 0\]
defined on some domain \(\Omega_X\) in X-space with boundary conditions
\[C_J(X,F^{(m-1)}) = 0\]
prescribed on boundary surfaces \(\nu_J(X) = 0\)

Condition (1.75) means that each boundary surface \(\omega_J(x) = 0\) is an invariant surface \(\nu_J(X) = 0\) of the infinitesimal generator
\[\xi^i(x) \frac{\partial}{\partial x_i}\]
which is the restriction of X to the x-space. The solution of (1.70) together with the BVP is an invariant solution
\[F = \Phi(\psi_1(x,u), \ldots, \psi_{N-1}(x,u))\]
The invariant solution \(u = f(x)\) defined by (1.70) satisfies
\[v[u - f(x)] = 0 \text{ when } u = f(x)\]
1.3 Invariance of a PDE

i.e.

\[ \xi^i(x) \frac{\partial u}{\partial x_i} = \phi(x, u) \]

**Theorem 1.3.15.** If the infinitesimal generator \( \nu \) is of the form

\[ \nu = \sum_{i=1}^{N} \xi^i(x) \frac{\partial}{\partial x_i} + h(x) u \frac{\partial}{\partial u} \]

then \( F \) is of the form

\[ F = \frac{u}{g(x)} \]

for a known function \( g(x) \) and hence an invariant solution arising from \( \nu \) is of the separated from

\[ u = g(x) \Phi(X) \]

for an arbitrary function \( \Phi(X) \) of

\[ (\psi_1(x, u), \ldots, \psi_{N-1}(x, u)) \]

If instead the PDE (1.70), again together with the BVP, admits an \( r \)-parameter Lie group of transformations with infinitesimal generators of the form

\[ v_i = \sum_j \xi^{ij}(x) \frac{\partial}{\partial x_j} + \phi^i(x, u) \frac{\partial}{\partial u}, \quad i = 1, \ldots, r \] (1.77)

the unique solution \( u = f(x) \) is an invariant solution satisfying

\[ v_i(u - f(x)) = 0 \quad i = 1, \ldots, r \]

**Theorem 1.3.16.** Suppose the BVP admits an \( r \)-parameter Lie group of transformations with infinitesimal generator of the form

\[ v_i = \sum_j \xi^{ij}(x) \frac{\partial}{\partial x_j} + h_i(x) u \frac{\partial}{\partial u}, \quad i = 1, \ldots, r \] (1.78)

Let \( R \) be the rank of the \( r \times n \) matrix

\[ T = \begin{bmatrix} \xi^{11}(x) & \cdots & \xi^{1n} \\ \vdots & \ddots & \vdots \\ \xi^{r1}(x) & \cdots & \xi^{rn}(x) \end{bmatrix} \] (1.79)

Let \( q = n - R \) and let \( Y_1(x), \ldots, Y_q(x) \) be a complete set of functionally independent invariants of (1.78) satisfying

\[ \xi^{ij}(x) \frac{\partial Y_l(x)}{\partial x_j} = 0, \quad i = 1, \ldots, r, \quad l = 1, \ldots, q \]
Let

\[ F = \frac{u}{g(x)} \]  
(1.80)

with \( g \) a known function be an invariant of (1.78) satisfying

\[ \nu_i[w] = 0, \quad i = 1, \ldots, r \]

Then the BVP reduces to a BVP with \( q = N - r \) independent variables \( Y = (Y_1, \ldots, Y_q) \) and dependent variable (1.80). The solution of the BVP is an invariant solution of separated form

\[ u = g(x)\Phi(Y) \]

where the function \( \Phi(Y) \) is to be determined.

### 1.4 Construction of mapping relating differential equations

In section 1.3 we considered the construction of the solution of a given PDE \( F(x, u^{(m)}) = 0 \) from other known solution of the PDE or as invariant solution.

In the present section we exploit a different approach. We will in fact use the infinitesimal transformation, and in particular the Lie algebra, to construct a transformation which maps a given PDE into another differential equation, the target DE, in the sense that any solution of the given DE is mapped into a solution of the target DE.

We will treat in particular only one-to-one mapping. This means that the map must establish a one-to-one correspondence between infinitesimal generators of the given and the target PDE. To be more precise it is necessary that any Lie algebra of the infinitesimal generators of the given PDE to be isomorphic to a lie algebra of the infinitesimal generators of the target PDE. In particular such a map is invertible. Anyway non invertible mapping can be constructed as well. Notation, if not otherwise specified, is as in the previous sections.

This procedure will be use in section 3.2 and in appendix C.1.

For a deeper treatment of the topic we refer to both [BK89; BCA10].

#### 1.4.1 Mapping of infinitesimal generators

Let us consider a PDE of the form

\[ F(x, u^{(m)}) = 0 \]  
(1.81)

with \( N \) independent variables \( x = (x_1, \ldots, x_N) \) and a single dependent variable \( u \). As usual the whole procedure can be extended to the general case of a system of DE’s with \( N \) dependent variables and \( q \) dependent variables. For such a problem we refer to [BK89; BCA10].

We will use the following notation:

\( G_x \) : the group of all admitted continuous transformation;
1.4 Construction of mapping relating differential equations

\( g_x \): the Lie algebra of \( G_x \);

\( H_x \): a subgroup of continuous transformations \( H_x \subset G_x \);

\( h_x \): the Lie algebra of \( H_x \), \( h_x \subset g_x \);

\( \exp(\epsilon v)^{(x)} \): a one-parameter transformations \( \exp(\epsilon v)^{(x)} \in H_x \);

\( v \): the infinitesimal generator of \( \exp(\epsilon v)^{(x)} \), \( v \in h_x \) of the form

\[
    v = \sum_i \xi^i x_i + \nu^x \partial_u.
\]

We consider the target PDE

\[
P(y, v^{(m)}) = 0 \tag{1.82}
\]

with again \( N \) independent variables \( y = (y_1, \ldots, y_N) \) and a single dependent variable \( v \). We will in particular use the following notation:

\( G_y \): the group of all admitted continuous transformation;

\( g_y \): the Lie algebra of \( G_y \);

\( H_y \): a subgroup of continuous transformations \( H_y \subset G_y \);

\( h_y \): the Lie algebra of \( H_y \), \( h_y \subset g_y \);

\( \exp(\epsilon v)^{(y)} \): a one-parameter transformations \( \exp(\epsilon v)^{(y)} \in H_y \);

\( w \): the infinitesimal generator of \( \exp(\epsilon v)^{(y)} \), \( w \in h_y \) of the form

\[
    w = \sum_i \xi^i y_i + \nu^y \partial_u.
\]

Let \( \mu \) denote a mapping which transforms any solution \( u = U(x) \) of equation (1.81) into a solution \( v = V(y) \) of equation (1.82). Of course this transformation need not to exist a priori and
even if it does it is unknown. Our goal is to establish a method that allows us to retrieve such a transformation. We must in particular restrict \( \mu \) to a specific mapping of the form

\[
\begin{align*}
  y &= \phi(x, u, \ldots, u^{(l)}) \\
  v &= \psi(x, u, \ldots, u^{(l)})
\end{align*}
\]  

(1.83)

We will denote by \( \mathcal{M}_l \) the class of mapping of the form (1.83) which depends on at most the \( l \)-th derivatives of \( u \). In particular any map \( \mu \in \mathcal{M}_m \) induces an action as in following diagram

\[
\begin{array}{ccc}
  F(x, u^{(m)}) & \xrightarrow{\exp(\epsilon v^{(x)})} & F(\tilde{x}, \tilde{u}^{(m)}) \\
  \mu \downarrow & & \downarrow \mu \\
  P(y, v^{(m)}) & \xrightarrow{\exp(\epsilon v^{(y)})} & P(\tilde{y}, \tilde{v}^{(m)})
\end{array}
\]  

(1.84)

The relationship between \( G_x, H_x, G_y, H_y \) and \( \mu \) is illustrated in figure 1.7.

The mapping \( \mu \) must take any symmetry \( \exp(\epsilon v^{(x)}) \) into a symmetry \( \exp(\epsilon v^{(y)}) \) such that the composition transformations

\[
\mu \circ \exp(\epsilon v^{(x)})(x) \quad \text{and} \quad \exp(\epsilon v^{(y)})(y) \circ \mu
\]  

(1.85)

yield the same action on the \( (x, u) \) space.

Specifically

\[
\mu \circ \exp(\epsilon v^{(x)})(x) = \mu(\tilde{x}, \tilde{u}) = \left( \phi(\tilde{x}, \tilde{u}, \ldots, \tilde{u}^{(m)}), \psi(\tilde{x}, \tilde{u}, \ldots, \tilde{u}^{(m)}) \right)
\]  

(1.86)

with

\[
\begin{align*}
  \tilde{x} &= x + \epsilon \xi^x(x, u, \ldots, u^{(m)}) + O(\epsilon^2), \\
  \tilde{u} &= u + \epsilon \nu^x(x, u, \ldots, u^{(m)}) + O(\epsilon^2), \\
  \tilde{u}^{(j)} &= u^{(j)} + \epsilon \nu^{x(j)}(x, u, \ldots, u^{(m+j)}) + O(\epsilon^2), \quad j = 1, \ldots, l
\end{align*}
\]  

(1.87)

On the other hand

\[
\exp(\epsilon v^{(y)})(y) \circ \mu(x, u) = \exp(\epsilon v^{(y)})(y, v) = (\tilde{y}, \tilde{v}) = \left( \phi(\tilde{x}, \tilde{u}, \ldots, \tilde{u}^{(m)}), \psi(\tilde{x}, \tilde{u}, \ldots, \tilde{u}^{(m)}) \right)
\]  

(1.88)

with

\[
\begin{align*}
  \tilde{y} &= y + \epsilon \xi^y(y, v, \ldots, v^{(m)}) + O(\epsilon^2), \\
  \tilde{v} &= v + \epsilon \nu^y(y, v, \ldots, v^{(m)}) + O(\epsilon^2)
\end{align*}
\]  

(1.89)

such that

\[
\begin{align*}
  \xi^y(y, v, \ldots, v^{(m)}) &= \xi^y(\phi, \psi, \ldots, \psi^{(m)}), \\
  \nu^y(y, v, \ldots, v^{(m)}) &= \nu^y(\phi, \psi, \ldots, \psi^{(m)})
\end{align*}
\]  

(1.90)
with
\[
\begin{align*}
\phi &= \phi(x, u, \ldots, u^{(l)}) \\
\psi &= \psi(x, u, \ldots, u^{(l)}) \\
\psi^{(j)} &= \psi^{(j)}(x, u, \ldots, u^{(l+j)}) \quad j = 1, \ldots, m
\end{align*}
\] (1.91)

Equating now the $O(\epsilon^2)$ terms in equation (1.86) and (1.88) we have, written in compact form, the necessary conditions that the mapping $\mu$ has to satisfy
\[
\begin{align*}
(v^{(m)})_\phi &= w_y, \\
(v^{(m)})_\psi &= w_u
\end{align*}
\] (1.92)

Figure 1.7: The mapping $\mu$

Conditions (1.92) play the essential role in all the algorithm we develop in the present section. In fact in order to such a transformation from the given PDE to the target PDE to exists, it is necessary that each infinitesimal generator $v \in h_x$ corresponds to some infinitesimal generator $w \in h_y$. Moreover the components $(\phi, \psi)$ of the transformation $\mu$ must satisfy conditions (1.92). These conditions play in a sense the same role the determining equations play in the determination of the admitted symmetries of a given PDE. Basically they reduce the nature of the dependence of the mapping $\mu$. In particular without such a conditions it is often impossible (and anyway particularly boring) to determine whether or not there exist a mapping $\mu \in M_l$. 
1.4.2 Theorems on invertible mapping

**Theorem 1.4.1.** Let $G_i$ be Lie groups and $g_i$ the corresponding Lie algebras, $i = 1, 2$. Then $g_1$ and $g_2$ are isomorphic iff $G_1$ and $G_2$ are locally analytically isomorphic.

In the special case of one-to-one (invertible) mapping from a given PDE to a target PDE the class of possible mappings $\mathcal{M}_1$ is predetermined. In particular for the case on a single dependent variable, as shown in the following theorem, we have that $l = 1$ i.e. $\mu \in \mathcal{M}_1$.

**Theorem 1.4.2** (Backlund(1876)). Let us consider a PDE with a single dependent variable $u$. Then a mapping $\mu$ defines an invertible mapping from $(x, u, \ldots, u^{(p)})$-space to $(y, v, \ldots, v^{(p)})$-space for any fixed $p$, iff $\mu$ is a one-to-one contact transformation of the form:

$$
\begin{align*}
    y &= \phi(x, u, u^{(1)}) \\
    v &= \psi(x, u, u^{(1)}) \\
    v^{(1)} &= \psi^{(1)}(x, u, u^{(1)})
\end{align*}
$$

(1.93)

Let us further notice that if $\phi$ and $\psi$ are independent of $u^{(1)}$ then defines a point transformation. A result for more that one dependent variable does exist but will not be treated in this thesis. For the interested reader we refer to [BK89; BCA10].

1.4.3 Lie’s classical results

We now recall Lie’s classical results on classification of second order linear partial differential equations. In the case of a parabolic equation it reads as follows.

**Theorem 1.4.3.** Let us consider the family of linear parabolic equations

$$
P(x, t)u_t + Q(x, t)u_x + R(x, t)u_{xx} + S(x, t)u = 0, \quad P \neq 0, \quad R \neq 0
$$

(1.94)

The principal Lie algebra of equation (1.94) is spanned by the generators of trivial symmetries

$$
\begin{align*}
    v_1 &= u \partial_u \\
    v_\alpha &= \alpha(x, t) \partial_u
\end{align*}
$$

(1.95)

Thus any equation of the form (1.94) can be reduced to an of equation of the form

$$
v_\tau = v_{yy} + Z(y, \tau) v
$$

(1.96)

by Lie’s equivalence transformation

$$
y = \alpha(x, t), \quad \tau = \beta(t), \quad v = \gamma(x, t)v, \quad \alpha_x \neq 0, \quad \beta_t \neq 0
$$

(1.97)

\[2\] It can be proved that $l = 0$ i.e. $\mathcal{M}_0$.\]
The group of dilations generated by the operator $v_1$ reflects the homogeneity of Equation (1.94), while the infinite group with the operator $v_\alpha$ represents the linear superposition principle for Equation (1.94).

If equation (1.94) admits an extension of the principal Lie algebra by one additional operator, namely

$$v_2 = \partial_t$$

then the original equation reduces to the form

$$v_\tau = v_{yy} + Z(y)v$$  \hfill (1.98)

If furthermore $g$ admits an extension by three additional operator, namely

$$\begin{cases} 
v_1 = \partial_t \\
v_2 = 2\tau \partial_\tau + y \partial_y \\
v_3 = \tau^2 \partial_\tau + \tau y \partial_y - \left(\frac{1}{2} y^2 + \frac{1}{2} \tau\right) v \partial_v \end{cases}$$  \hfill (1.99)

then equation (1.94) reduces to the form

$$v_\tau = v_{yy} + A \frac{y}{y^2} v$$  \hfill (1.100)

If $g$ admits five additional operator

$$\begin{cases} 
v_1 = \partial_t \\
v_2 = 2\tau \partial_\tau + y \partial_y \\
v_3 = \tau^2 \partial_\tau + \tau y \partial_y - \left(\frac{1}{2} y^2 + \frac{1}{2} \tau\right) v \partial_v \\
v_4 = \partial_y \\
v_5 = 2\tau \partial_y - y v \partial_v \end{cases}$$  \hfill (1.101)

then equation (1.94) can be mapped in the heat equation

$$v_\tau = v_{yy}$$  \hfill (1.102)

Equations (1.98), (1.100) and (1.102) provide the canonical forms of all linear parabolic second-order equations (1.94) that admit non-trivial symmetries.

1.5 The Heat Equation

We will now apply all the theory stated to the one-dimensional (in space) heat equation.

Theorem 1.2.15 together with the prolongation formulae (1.41) and (1.42) give a rigorous procedure to evaluate a symmetry group of a given partial differential equation.

Let us now finally give a concrete example of how the Lie group and the local transformation that
Let us thus consider the one-dimensional heat equation

\[ u_t = u_{xx} \]  

(1.103)

We can identify the heat equation with the linear subvariety in \( X \times U^{(2)} \) determined by the vanishing of \( P(x, t, u^{(2)}) = u_t - u_{xx} \). Let

\[ v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u} \]  

(1.104)

be a vector field on \( X \times U \). We are willing to evaluate \( \xi, \tau, \phi \) such that the corresponding one-parameter group \( \exp(\epsilon v) \) is a symmetry group of the heat equation. By theorem 1.2.15, using the second prolongation

\[ \text{pr}^{(2)} v = v + \phi^x(x, t, u) \frac{\partial}{\partial u_x} + \phi^t(x, t, u) \frac{\partial}{\partial u_t} + \phi^u(x, t, u) \frac{\partial}{\partial u_u} \]  

(1.105)

Applying therefore \( \text{pr}^{(2)} v \) to (1.119) we obtain

\[ \phi^t = \phi^{xx} \]  

(1.106)

From (1.42) we can evaluate

\[ \phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} = D_t \phi - u_x D_t \xi - u_t D_t \tau = \]

\[ \phi_t - \xi_x u_x + (\phi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2 \]

and

\[ \phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \]

\[ = D_x^2 \phi - u_x D_x^2 \xi - u_t D_x^2 \tau - 2u_x D_x \xi - 2u_t D_x \tau = \]

\[ = \phi_{xx} + (2\phi_{xx} - \xi_{xx}) u_x - \tau_{xx} u_t + (\phi_{uu} - 2\xi_{xx}) u_x^2 - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 + \]

\[ - \tau_{uu} u_x^2 u_t + (\phi_u - 2\xi_x) u_{xx} - 2\tau_x u_{xx} - 3\xi_u u_x u_{xx} - \tau_u u_x u_{xx} - 2\tau_u u_x u_{xt} \]

(1.107)

Substituting now this two equation into (1.106) and equating \( u_t = u_{xx} \) we find the determining
equation for the symmetric group of the heat equation to be

\[ u_{xt} u_x (-2\tau_t) \quad (1.108a) \]
\[ u_{xt} (-2\tau_x) \quad (1.108b) \]
\[ u_{xx}^2 (\tau_u - \tau_u) \quad (1.108c) \]
\[ u_{xx} u_x^2 (-\tau_{uu}) \quad (1.108d) \]
\[ u_{xx} u_x (\xi_u - 2\tau_{ux} - 2\zeta_u) \quad (1.108e) \]
\[ u_{xx} (\tau_t - \phi_u - \tau_{xx} + \phi_u - 2\xi_x) \quad (1.108f) \]
\[ u_x^3 (-\xi_{uu}) \quad (1.108g) \]
\[ u_x^2 (-2\xi_{xu} + \phi_{uu}) \quad (1.108h) \]
\[ u_x (\xi_t + 2\phi_{xu} - \xi_{xx}) \quad (1.108i) \]
\[ 1(-\phi_t + \phi_{xx}) = 0 \quad (1.108j) \]
\[ u_{xx}^3 (-\xi_{uu}) \quad (1.108k) \]

We have now to equate each monomial to 0. First of all let us notice that (3.34a) and (3.34b) requires that \( \tau \) depends only on \( t \). Then (3.34e) implies that \( \xi \) does not depend on \( u \) and (1.108f) implies \( \xi(x,t) = \frac{1}{2}\tau_t x + \sigma(t) \). Further from (1.108h) we can deduce that \( \phi \) is linear in \( u \) i.e. \( \phi(x,t,u) = \beta(x,t) u + \alpha(x,t) \) for some functions \( \alpha \) and \( \beta \).

For what it concerns (1.108i) \( \xi_t = -2\beta_x \) so that \( \beta = -\frac{1}{8}\tau_{tt} x^2 - \frac{1}{2}\sigma_t x + \rho(t) \). Finally from (1.108j) we can say that both \( \alpha \) and \( \beta \) are solutions for the heat equation i.e.

\[ \alpha_t = \alpha_{xx} \quad \beta_t = \beta_{xx} \]

Therefore we obtain

\[ \tau_{ttt} = 0 \quad \sigma_{tt} = 0 \quad \rho_t = -\frac{1}{4}\tau_{tt} \]

We can thus conclude that the most general form of the coefficients of the infinitesimal symmetry of the heat equation are

\[
\begin{align*}
\xi &= c_1 + c_4 x + 2c_3 t + 4c_6 x t \\
\tau &= c_2 + 2c_4 t + 4c_6 t^2 \\
\phi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2) u + \alpha(x,t)
\end{align*}
\]

Therefore we have found that the Lie algebra of infinitesimal symmetries of the heat equation is
spanned by
\[
\begin{align*}
    v_1 &= \partial_x \\
    v_2 &= \partial_t \\
    v_3 &= x\partial_x + 2t\partial_t \\
    v_4 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \\
    v_5 &= 2t\partial_x - xu\partial_u \\
    v_6 &= u\partial_u
\end{align*}
\]

together with the infinite dimensional subalgebra
\[
v_\alpha = \alpha(x,t)\partial_u
\]
where \(\alpha\) is an arbitrary solution of the heat equation.

The commutator table is therefore
\[
\begin{array}{cccccc}
    & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
    v_1 & 0 & 0 & v_1 & v_5 & -\frac{1}{2}v_6 & 0 \\
    v_2 & 0 & 0 & 2v_2 & v_3 - \frac{1}{2}v_6 & v_1 & 0 \\
    v_3 & -v_1 & -2v_2 & 0 & 2v_4 & v_5 & 0 \\
    v_4 & -v_5 & -v_3 + \frac{1}{2}v_6 & -2v_4 & 0 & 0 & 0 \\
    v_5 & \frac{1}{2}v_6 & -v_1 & -v_1 & 0 & 0 & 0 \\
    v_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The one-parameter group \(G_i\) generated by the \(v_i\) are thus given exponentiating the infinitesimal generator, i.e. the transformed points are \(\exp(\epsilon v_i)(x,t,u) = (\tilde{x}, \tilde{t}, \tilde{u})\). Applying (1.48) we find
\[
\begin{align*}
    G_1 &: (x + \epsilon,t,u) \\
    G_2 &: (x,t + \epsilon,u) \\
    G_3 &: (x,t,e^\epsilon u) \\
    G_4 &: (e^\epsilon x,e^{2\epsilon}t,u) \\
    G_5 &: (x + 2\epsilon t,u \exp(-\epsilon x - \epsilon^2 t)) \\
    G_6 &: \left(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, u\sqrt{1-4\epsilon} \exp\left(-\frac{\epsilon x^2}{1-4\epsilon t}\right)\right) \\
    G_\alpha &: (x,t,u + \epsilon \alpha(x,t))
\end{align*}
\]

Now from (1.30) and from the fact that each \(G_i\) is a symmetric group we can say that whenever
1.5 The Heat Equation

\( u = f(x, t) \) is a solution of the heat equation so are the functions

\[
\begin{align*}
    u^{(1)} &= f(x - \epsilon, t) \\
    u^{(2)} &= f(x, t - \epsilon) \\
    u^{(3)} &= e^\epsilon f(x, t) \\
    u^{(4)} &= f(e^{-\epsilon}x, e^{-2\epsilon}t) \\
    u^{(5)} &= e^{-\epsilon x + \epsilon^2 t} f(x - 2\epsilon t, t) \\
    u^{(6)} &= \frac{1}{\sqrt{1 - 4\epsilon t}} \exp \left( -\frac{\epsilon x^2}{1 + 4\epsilon t} \right) \frac{d}{d(1 + 4\epsilon t)} f \left( \frac{x}{1 + 4\epsilon t}, \frac{t}{1 + 4\epsilon t} \right) \\
    u^{(\alpha)} &= f(x, t) + \epsilon \alpha(x, t)
\end{align*}
\]

We would like now to recover a solution to the heat equation using the algebra just found.

Invariant solutions \( u = f(x, t) \) of the heat equation corresponding to \( v_6 \) satisfy

\[
4xt \frac{\partial f}{\partial x} + 4t^2 \frac{\partial f}{\partial t} = -(x^2 + 2t) f \quad (1.109)
\]

We can find a solution by solving the characteristic equations

\[
\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)} \quad (1.110)
\]

We thus have two invariants:

\[
X_1 = \frac{x}{t} \quad v = \sqrt{t} e^{\frac{x^2}{4t}} u \quad (1.111)
\]

Therefore the solution of (1.109) is defined by the invariant form

\[
\sqrt{t} e^{\frac{x^2}{4t}} u = \phi \left( \frac{x}{t} \right) \quad (1.112)
\]

or solving for \( u \)

\[
u = f(x, t) = \frac{1}{t} e^{-\frac{x^2}{4t}} \phi(\zeta) \quad (1.113)\]

where the similarity variable is

\[
\zeta = \frac{x}{t} \quad (1.114)
\]

Substituting now (1.113) into the heat equation we obtain

\[
\phi''(\zeta) = 0 \quad (1.115)
\]

so that the invariant solutions resulting from \( v_6 \) are

\[
u = f(x, t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left( C_1 + C_2 \frac{x}{t} \right) \quad (1.116)
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

We have already found the one-parameter family of solutions \( u = f(x, t; \epsilon) \) generated by \( v_6 \) to be
\[ G_\delta : \left( \frac{x}{1 - 4\epsilon t}, \frac{t}{1 - 4\epsilon t}, u \sqrt{1 - 4\epsilon t} \exp \left( -\frac{\epsilon x^2}{1 - 4\epsilon t} \right) \right) \]  
(1.117)

Then

\[ u = f(x, t; \epsilon) = \frac{1}{\sqrt{1 - \epsilon t}} \exp \left( \frac{\epsilon x^2}{4(1 - \epsilon t)} \right) f \left( \frac{x}{1 - \epsilon t}, \frac{t}{1 - \epsilon t} \right) \]  
(1.118)

We want to use the method explained to recover fundamental solutions of the heat equation via invariance method for finite and finite domain.

We will once again consider the one-dimensional heat equation

\[ u_t = u_{xx} \]  
(1.119)

defined on the domain

\[ \begin{cases} 
  t > 0 \\
  a < x < b 
\end{cases} \]
Recall that the determining equations for the 6-parameter Lie groups of transformations are

\[
\begin{align*}
\xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt \\
\tau &= c_2 + 2c_4 t + 4c_6 t^2 \\
\phi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2) u + \alpha(x, t)
\end{align*}
\]

(1.120)

From the invariance of \( t = 0 \) we have \( \tau(0) = 0 \) and therefore

\[ c_2 = 0 \]

If the domain in infinite, i.e. \( a = -\infty \) and \( b = \infty \) there is no further parameter reduction from the invariance of the boundary surfaces. If instead \( a \neq -\infty \) then the invariance of the surface \( x = a \) leads to \( \xi(a, t) = 0 \) for any \( t \) and hence we obtain

\[ c_1 = -c_4 a, \quad 2c_5 = -4c_6 a \]

Again if \( b \neq \infty \) the invariance of the surface \( x = b \) yields

\[ c_1 = -c_4 b, \quad 2c_5 = -4c_6 b \]

Therefore it is easy to see that if both \( a, b \neq \infty \), then \( c_1 = c_4 = c_5 = c_6 = 0 \) and thus there is no non trivial group admitted by the BVP for the heat equation.

However, as briefly mentioned in section 1.3.3, since we are dealing with a linear PDE, it not necessary that the Lie group of transformations leaves all the BVP invariants.

(i) Infinite domain \((a, b) = (-\infty, \infty)\).

Let us consider the problem

\[ u_t = u_{xx} \]

with boundary conditions

\[ u(\pm \infty, t) = 0, \quad t > 0; \quad u(x, 0) = \delta(x) \]

where \( \delta \) is the Dirac mass.

The infinitesimal 1.120 with \( c_2 = 0 \) is admitted by the BVP provided that

\[ \phi(x, 0)u(x, 0) = \xi(x, 0)\delta'(x) \]

when \( u(x, 0) = \delta(x) \) i.e.

\[ \phi(x, 0)\delta(x) = \xi(x, 0)\delta'(x) \]

The last equation is satisfied if

\[ \xi(0, 0) = 0 \quad \text{and} \quad \phi(0, 0) = -\frac{\partial \xi}{\partial x}(0, 0) \]
implying thus
\[ c_1 = 0, \quad c_3 = -c_4 \]

We have hence found that a three parameter Lie group of transformations leaves the BVP invariant.

Infinitesimal generators of this group are
\[
\begin{align*}
v_1 &= x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} \\
v_2 &= t \frac{\partial}{\partial x} - \frac{1}{2} xu \frac{\partial}{\partial u} \\
v_3 &= xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - \left[ \frac{x^2}{4} + \frac{t}{2} \right] u \frac{\partial}{\partial u}
\end{align*}
\]

The matrix (1.79) is now
\[
T = \begin{bmatrix} x & 2t \\ t & 0 \\ xt & t^2 \end{bmatrix}
\] (1.121)

T has rank two so that the number of independent variables reduces to 0. Note further that
\[ v_3 = \frac{1}{2} v_1 + xv_2 \]

Hence an invariant solution corresponding to \( v_1 \) and \( v_2 \) is also an invariance solution for \( v_3 \).

Let \( u = f(x_1) \) be an invariant solution corresponding to \( v_1 \) and \( v_2 \)
\[
v_1 \left[u - f(x_1)\right] = 0
\]
leads to the invariant of the form
\[
u = f(x_1) = \frac{1}{\sqrt{t}} F_1(\zeta_1)
\]
with similarity variable \( \zeta_1 = \frac{x_1}{\sqrt{t}} \).

The equation
\[
v_2 \left[u - f(x_1)\right] = 0
\]
leads to the invariant form
\[
u = f(x_1) = e^{-\frac{x_1^2}{4t}} F_2(\zeta_2)
\]
with similarity variable
\[ \zeta_2 = t \]

From the uniqueness of the solution of the BVP we have
\[
\frac{1}{\sqrt{t}} F_1(\zeta_1) = e^{-\frac{x_1^2}{4t}} F_2(\zeta_2)
\]
and therefore
\[ \sqrt{\zeta_2} F_2(\zeta_2) = e^{\frac{\zeta_2}{4}} F_1(\zeta_1) = \text{const} = c \]

Hence the solution of the BVP is
\[ u = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4t}} \]

The initial condition leads to
\[ c = \frac{1}{\sqrt{4\pi}} \]

From what already said it follows
\[ \mathbf{v}_3 \left[ u - \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \right] = 0 \]

(ii) Semi-infinite domain \((a, b) = (0, \infty)\).

Let us now instead consider the problem
\[ u_t = u_{xx} \]

with boundary conditions
\[ u(0, t) = 0, t > 0; \quad u(x, 0) = \delta(x - \hat{x}) \]

where \(\delta\) is the Dirac mass.

A three parameter Lie group of transformations is admitted by the BVP. The invariance of the second surface means that
\[ \phi(x, 0) u(x, 0) = \xi(x, 0) \delta'(x - \hat{x}) \]

when \(u(x, 0) = \delta(x - \hat{x})\) i.e.
\[ \phi(x, 0) \delta(x - \hat{x}) = \xi(x, 0) \delta'(x - \hat{x}) \]

Hence
\[ \xi(\hat{x}, 0) = 0 \quad \text{and} \quad \phi(\hat{x}, 0) = -\frac{\partial \xi}{\partial x}(\hat{x}, 0) \]

We thus must have
\[ c_4 = 0, \quad c_3 = \frac{c_0 \hat{x}^2}{4} \]

Thus the BVP admits infinitesimal generator
\[ \mathbf{v}_1 = x t \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + \left[ \frac{\hat{x}^2}{4} - \left( \frac{x^2}{4} + \frac{t}{2} \right) \right] u \frac{\partial}{\partial u} \]
The corresponding invariant solution has the invariant form

\[ u = f(x) = \frac{1}{\sqrt{t}} e^{-\frac{x^2 + \hat{x}^2}{4t}} F(\zeta) \]

where \( F(\zeta) \) is an arbitrary function of the similarity variable \( \zeta = \frac{x}{t} \). Substituting the solution so found in the heat equation (1.119) we find that \( F(\zeta) \) satisfies the ODE

\[ \frac{d^2 F(\zeta)}{d\zeta^2} = \frac{\hat{x}^2}{4} F(\zeta) \]

and hence

\[ u = f(x) = \frac{1}{\sqrt{t}} \left( Ce^{-\frac{(x-\hat{x})^2}{4t}} + De^{-\frac{(x+\hat{x})^2}{4t}} \right) \]

From the initial datum it follows that

\[ C = -D = \frac{1}{\sqrt{4\pi}} \]

leading to the well known solution of the heat equation.
Chapter 2

Fundamental solutions and transition densities

2.1 Preliminary results

We have previously empathized the role played by invariant solutions i.e. solution that are mapped into themselves by a given symmetry group. The observant reader should have noticed that we mentioned as well solutions that are mapped into other solutions, without requiring the solutions to be the same. This means we can construct a given solution from a trivial one. The method we will develop in this chapter exploit this kind of properties.

The idea that lies behind the following method is to produce an integral transform of a solution by applying a Lie symmetry to a trivial solution. We will use the Fourier and the Laplace transforms since they are among the most studied transformations and extensive tables does exist.

2.1.1 Group transformation of known solution

We will consider for the whole present chapter the Ito process $X = \{X_t : t \geq 0\}$ solution of the SDE

$$dX_t = f(X_t)dt + \sqrt{2\sigma X_t^\gamma}dW_t, \quad X_0 = x_0 > 0$$

(2.1)

where as usual $W = \{W_t : t \geq 0\}$ is a standard Brownian motion.

The process (2.1) can be link to the PDE of the form

$$u_t = \sigma x^\gamma u_{xx} + f(x)u_x - g(x)u$$

(2.2)

where $\sigma > 0$ and $\gamma$ a constant.

The present method to construct exact solution is based on the fact that a symmetry group
transfers any solutions of the equation into solution of the same equation. Namely, let
\[
\begin{align*}
\tilde{x} &= X_\varepsilon(x, u) \\
\tilde{u} &= U_\varepsilon(x, u)
\end{align*}
\] (2.3)
be a Lie group of transformations of equation
\[
F(x, u^{(m)}),
\] (2.4)
and let a function
\[
u = f(x)
\] (2.5)
solve equation (2.4). Since (2.3) is a symmetry transformation, the solution (2.5) can be also written
using the new variables
\[
\tilde{u} = f(\tilde{x})
\]
Replacing here \( \tilde{x}, \tilde{u} \) from equations (2.3), we get
\[
U_\varepsilon(x, u) = f(X_\varepsilon(u, x))
\]
Having solved this equation with respect to \( u \), we arrive at the following one-parameter family of new
solutions of equation (2.4)
\[
u = U_\varepsilon(x)
\]
Consequently, any known solution is a source of a multi-parameter class of new solutions provided
that the differential equation considered admits a multi-parameter symmetry group.
As an example see section ?? where the symmetric group for the heat equation is given.

2.1.2 Fundamental solutions

Investigation of initial value problems for parabolic linear partial differential equations (hyperbolic
as well but we will not consider the case in the thesis) can be reduced to the construction of a particular
solution with specific singularities known in the literature as fundamental solutions. What we are
doing here is just a brief introduction to fundamental solutions. For some general notions on
fundamental solutions for parabolic equations see appendix B.1.3 or for a more interested reader we
refer to Friedman [Fri64] and Evans [Eva10].

We now state the definition of fundamental solution for a given PDE. First of all a Cauchy problem
is a PDE together with a boundary condition as follows
\[
\begin{align*}
ut &= a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u, & x \in \Omega \subseteq \mathbb{R}^N \\
u(x, 0) &= \phi(x)
\end{align*}
\] (2.6)

**Definition 2.1.1.** A fundamental solution for the Cauchy problem (2.6) is a function \( p(x, y, t) \) such
that:
2.1 Preliminary results

- ∀ y fixed, \( p(x, y, t) \) is a solution of (2.6) on \( \Omega \times (0, T] \) for some \( T > 0 \);

- \( u(x, t) = \int_\Omega \phi(y)p(x, y, t)dy \) is a solution of (2.6).

The fundamental solution is thus the solution of the following Cauchy problem

\[
\begin{cases}
    u_t = \sigma \chi + f(x)u_x - g(x)u, & t < t_0 \\
    u(x, t)_{|t\to t_0} = \delta(x - y)
\end{cases}
\] (2.7)

where \( \delta(x - y) \) denotes the Dirac mass at the point \( y \). In general we will consider \( t_0 = 0 \). The connection between definition 2.1.1 and the Cauchy problem (2.7) can be seen via convolution.

The idea is to find a fundamental solution of equation (2.2). If the fundamental solution of this PDE satisfies certain conditions, then it gives the probability density function for the process \( X \).

What we will be doing is starting from (2.2), using the method previously exposed, we will find a basis for the Lie algebra. Therefore composing properly this basis we will find an infinitesimal generator such that the related symmetric group \( U_\epsilon(t, x) \) possesses certain properties.

Now let us suppose that (2.2) has a fundamental solution \( p(x, y, t) \), therefore the function

\[
u(x, t) = \int_\Omega \phi(y)p(x, y, t)dy
\] (2.8)

has to solve the initial value problem (2.6).

The main idea of the method is to connect (2.8) with the symmetric group \( U_\epsilon(t, x) \).

Taking \( u_0 = 1 \) stationary solution we will often be able to recover the transition density function of the process to be studied. Thus from definition 2.1.1, setting thus \( t = 0 \) and recalling that \( U_\epsilon(t, x) \) is as matter of a fact a solution of (2.2) we have the following

\[
\int_\Omega U_\epsilon(0, x)p(x, y, t)dy = U_\epsilon(t, x)
\] (2.9)

The key point is that we have to choose \( \epsilon \) such that the the family of solutions \( U_\epsilon(x) \) reduces to a Laplace or a Fourier transform when setting \( t = 0 \). This will allow us to invert (2.9) and then to recover the transition density. The inversion does not imply \( p(x, y, t) \) to be a transition density. However we will see that some theorems exist such that this is true.

Let us now give an example in order to make the method more clear.

**Example 2.1.1.** We have previously found the basis for the Lie symmetry algebra for the one-dimensional heat equation.

We have in fact seen in section ?? that if \( u(x, t) \) solves the heat equation so does \( u^{(5)} = \exp(-\epsilon x + \epsilon^2 t)u(x - 2\epsilon t, t) \). Taking now as said \( u_0 = 1 \) we will have

\[
\int_{\mathbb{R}} e^{-\epsilon y} p(t, x - y)dy = \exp(-\epsilon x + \epsilon^2 t)
\]

This is as we wanted a Laplace transform and we can thus recover \( p(t, x - y) \) inverting it.
2. Fundamental solutions and transition densities

2.1.3 Integrating symmetries

Suppose we are dealing with a PDE of the form

\[ F(x, u^{(m)}) = \sum_{|\alpha| \leq m} a_{\alpha}(x) D^{\alpha}, \quad x \in \Omega \subseteq \mathbb{R}^N \]  

(2.10)

with \( \alpha = (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{N}, |\alpha| = \alpha_1 + \cdots + \alpha_N \) and \( D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \).

Let now \( G \) be a one parameter group of transformations, generated by the vector field

\[ v = \sum_{i=1}^{N} \xi(x) \partial_{x_i} + \phi(x, u) \partial_u \]

As usual we will denote the action of \( G \) on a solution \( u \) by \( \exp(\epsilon v)(u(x)) = (\exp \epsilon v)u(x) = U_\epsilon(x) \). Thus by the Lie symmetry property it exists an interval \( I \subseteq \mathbb{R} \) containing the identity \( e = 0 \) such that \( U_\epsilon(x) \) is a continuous one parameter family of solutions of (2.10). From continuity and linearity we have the following

**Lemma 2.1.2.** Suppose \( U_\epsilon(x) \) is a continuous one parameter family of solution of the PDE (2.10). Suppose furthermore the \( \varphi : I \to \mathbb{R} \) is a function with sufficiently rapid decay. Then

\[ u(x) = \int_I \varphi(\epsilon) U_\epsilon(x) d\epsilon \]

is a solution of (2.10). Furthermore if the PDE is time dependent we can define \( U_\epsilon(x, t) \) the family of symmetry solutions to have

\[ u(x, t) = \int_I \varphi(\epsilon) U_\epsilon(x, t) d\epsilon \quad \text{and} \quad u(x, 0) = \int_I \varphi(\epsilon) U_\epsilon(x, 0) d\epsilon. \]

Further

\[ \frac{d^n U_\epsilon(x)}{d\epsilon^n} \]

is also a solution for all \( n = 0, 1, 2, \ldots \).

**Proof.** It is enough to differentiate under the integral. If the interval \( I \) is unbounded then the function \( \varphi \) must be of compact support in order to have the convergence of the integral. Eventually the lemma is proved noticing that the PDE does not depend on \( \epsilon \) so that the order of differentiability may be reversed.

2.1.4 Fundamental solutions as transition densities

In this section we will provide a link between SDE and PDE. In particular we will show how to use the previous result studying a PDE. The importance of fundamental solution is that they are the link from analysis to probability theory (and viceversa).
Let us consider an Ito diffusion \( X = \{ X_t : t \geq 0 \} \) which satisfies the SDE
\[
\begin{align*}
    dX_t &= \mu(X_t, t)dt + \sigma(X_t, t)dW_t \\
    X_0 &= x
\end{align*}
\] (2.11)
where as usual \( W_t \) denotes a standard Wiener process. Let us assume that \( \mu \) and \( \sigma \) satisfy certain conditions such that a unique solution does exist. Then
\[
u(x, t) = \mathbb{E}[\phi(X_t)] := \mathbb{E}[\phi(X_t)|X_0 = x]
\] (2.12)
is a solution of the Cauchy problem
\[
\begin{align*}
    u_t &= \frac{1}{2} \sigma^2(x, t)u_{xx} + \mu(x, t)u_x, \\
    u(x, 0) &= \phi(x)
\end{align*}
\] (2.13)
The PDE (2.13) is known as Kolmogorov forward equation. We will not enter in this thesis the discussion about well-posedeness and ill-posedeness of parabolic equation. We will just recall that when dealing with PDE’s in financial mathematics describing the evolution of the value of a given option, the datum is prescribed at maturity. This implies that the forward parabolic equation is ill-posed and thus a change of variable \( \tau = T - t \) with \( T \) maturity date has to be made.

Back to the beginning, if \( p(x, y, t) \) is an appropriate fundamental solution of (2.13), we can compute the expectation according to definition 2.1.1 and equation (2.12) \( u(x, t) = \mathbb{E}_x[\phi(X_t)] = \int_\Omega \phi(y)p(x, y, t)dy \). In a probability context the fundamental solution is known as transition density function. Of course in order \( p(x, y, t) \) to be a density function we have to require \( \int_\Omega p(x, y, t)dy = 1 \).

Fundamental solutions are not unique. The PDE (2.13) may have more than one fundamental solution, and of course only one of them can be the desired transition density function. The strength of the Lie algorithm is that we are able to recover the fundamental solution that gives us the probability density function (pdf).

**Example 2.1.2.** Let us consider the diffusion \( X = \{ X_t : t \geq 0 \} \) which satisfies the SDE
\[
\begin{align*}
    dX_t &= \frac{2aX_t}{2 + ax} + \sqrt{2X_t}dW_t \\
    X_0 &= x > 0, \quad a > 0
\end{align*}
\] (2.14)
Since the drift is Lipschitz continuous for \( a > 0 \) and \( \sqrt{x} \) is Holder continuous by the Yamada-Watanabe theorem the SDE (2.14) has a unique strong solution. We thus look for a fundamental solution of
\[
    u_t = xu_{xx} + \frac{2ax}{2 + ax}u_x
\] (2.15)
It can be shown that equation (2.15) has a fundamental solution of the form
\[
    q(x, y, t) = \frac{2 + ay}{t(2 + ax)} \sqrt{\frac{2}{y}} \exp\left\{ -\frac{x + y}{t} \right\} I_1\left( \frac{2\sqrt{xy}}{t} \right)
\] (2.16)
The problem is that the fundamental solution (2.16) cannot be the pdf. In fact let us notice that
\[
\int_0^\infty q(x, y, t) dy = 1 - \exp\left\{-\frac{x^2 t}{2 + ax}\right\} \neq 1
\]
In fact it has been shown in [CP04] that the transition probability function for (2.14) is
\[
p(x, y, t) = \exp\left\{-\frac{x + y}{t(2 + ax)}\right\} \left[\sqrt{\frac{x}{y}}(2 + ay)I_1\left(\frac{2\sqrt{xy}}{t}\right) + t\delta(y)\right]
\]
In general in order to obtain a pdf it is often necessary to include additional terms which involve
generalized function such as the Dirac mass. The method we develop in the present chapter gives us
these additional terms that cannot otherwise be found.

A more general result that emphasize once more the connection between SDE and parabolic PDE
is the Feynman-Kac theorem.

**Theorem 2.1.3** (Feynman-Kac). Let \( u \in C^2(S_T) \cap C(\bar{S}_T) \) be a solution of the Cauchy problem
\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 u_{xx} + \mu u_x - ru + f &= 0, \quad \text{in } S_T := [0, T) \times \mathbb{R}^m \\
u(x, 0) &= \phi(x)
\end{align*}
\]
Let \( X = X_t : t \geq 0 \) be an Ito diffusion process which satisfies the SDE
\[
\begin{align*}
dX_t &= \mu(X_t, t)dt + \sigma(X_t, t)dW_t \\
X_0 &= x
\end{align*}
\]  
(2.17)

Let us now assume that:

(i) \( \mu \) and \( \sigma \) have at most linear growth in \( x \);

(ii) \( \forall (t, x) \in S_T, \exists \) a solution \( X \) of the SDE (2.17);

Let us assume furthermore that at least one of following holds:

1. \( \exists \) two constants \( M, p > 0 \) :

\[
|u(t, x)| + |f(t, x)| \leq M(1 + |x|^p) \quad (t, x) \in S_T
\]

2. the matrix \( \sigma \) is bounded and \( \exists \) two constants \( M, \alpha > 0 \), with \( \alpha \) small enough :

\[
|u(t, x)| + |f(t, x)| \leq M(\exp^\alpha |x|^2) \quad (t, x) \in S_T
\]
Then \( \forall (t, x) \in S_T \) we have the following formula

\[
u(x, t) = \mathbb{E}_{x, t} \left[ \exp \left( - \int_t^T r(u, X_u)du \right) \phi(X_T) + \int_t^T \exp \left( \int_u^T r(z, X_z)dz \right) f(u, X_u)du \right]
\]  

\[(2.18)\]

### 2.2 The square root process

The present section is devoted to some general results that give us the explicit formulation of solution for a wide class of PDE’s. We will proceed as explained in the previous section. Eventually we will use these results in chapter 3.

#### 2.2.1 General results

As said we are trying to invert via Laplace transform a fundamental solution. Such a fundamental solution in in general a distribution (or generalized function). Let us first of all introduce the generalized Laplace transform for distributions.

**Definition 2.2.1 (Generalized Laplace transform).** Let \( f : [0, \infty) \to \mathbb{R} \) be Lebesgue integrable and be of suitable slow growth. The generalized Laplace transform of \( f \) is the function

\[
F_{\gamma}(\lambda) = \int_0^\infty f(y)e^{-\lambda y^{2-\gamma}}dy
\]

where \( \lambda > 0 \) and \( \gamma \neq 2 \).

This transform is usually inverted setting \( z = y^{2-\gamma} \) in order to deal with a Laplace transforms.

**Proposition 2.2.2.** The PDE

\[
u_t = \sigma x^\gamma u_{xx} + f(x)u_x - g(x)u, \quad \gamma \neq 2
\]

has a Lie algebra of symmetries which is at least four dimensional iff, for a given function \( g \), the function \( h(x) = x^{1-\gamma} f(x) \), is a solution of one of the following families of Riccati equations:

\[
\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma Ax^{2-\gamma} + B
\]

\[
\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{Ax^{4-2\gamma}}{2(2-\gamma)^2} + \frac{Bx^{2-\gamma}}{2-\gamma} + C
\]

\[
\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{Ax^{4-2\gamma}}{2(2-\gamma)^2} + \frac{Bx^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{Cx^{2-\gamma}}{2-\gamma} - K
\]

with \( K = \frac{1}{16}(\gamma - 4)\sigma^2 \), and \( A,B \) and \( C \) arbitrary constants.

Symmetry solutions can be found for \( h \) that satisfies any of the three previous equations. Anyhow we will state only results concerning the first and second Riccati equation.
Thus we can find fundamental solution inverting a generalized Laplace transform. we now state an important result that we will need when trying to recover fundamental solutions.

**Proposition 2.2.3.** When \( n \) is a non negative integer we have

\[
L^{-1} \left[ \lambda^n e^{\frac{k}{2}} \right] = \sum_{l=0}^{n} \frac{k^l}{l!} \delta^{(n-l)}(y) + \left( \frac{k}{y} \right)^{\frac{n+1}{2}} I_{n+1}(2\sqrt{ky})
\]

where \( L \) is the Laplace transform, \( \delta(y) \) is the Dirac delta and \( I_n \) is a modified Bessel function of the first Kind.

Hopefully we are now able to recover fundamental solutions for certain PDE’s. We are still not able to say anything about this fundamental solution to be a probability transition density. We will now look for certain condition under which we can say that the fundamental solution found is in fact a transition density.

In general we are able to recover a pdf in the case of \( g = 0 \). For the more general situation of \( g \neq 0 \) some problem may arise.

**Corollary 2.2.4.** Suppose that the conditions of 2.2.10 hold and that \( g = 0 \). Let us take the stationary solution \( u_0 = 1 \). The resulting fundamental solution \( p(x, y, t) \) has the property that

\[
\int_0^\infty p(x, y, t) dy = 1.
\]

This result is still not enough to say that the fundamental solution is a transition density. In order to do this we need the following results.

**Proposition 2.2.5.** Let \( X = \{ X_t : t \geq 0 \} \) be an Ito process solution of

\[
dX_t = f(X_t) dt + \sqrt{2\sigma X_t} dW_t
\]

with \(|f(x)| \leq Ke^{ax} \) measurable. It therefore exists \( T : u(x, t, \lambda) = \mathbb{E}_x[e^{-\lambda X_t}] \) is the unique solution of

\[
u_t + \lambda^2 \sigma u + \lambda \mathbb{E}_x[f(X_t)e^{-\lambda X_t}] = 0
\]

subject to \( u(x; 0, \lambda) = e^{-\lambda x} \) for \( 0 \leq t < T, \lambda > a. \)

**Corollary 2.2.6.** Let \( X = \{ X_t : t \geq 0 \} \) be the unique strong solution of the SDE

\[
dX_t = f(X_t) dt + \sqrt{2\sigma X_t} dW_t, \quad X_0 = x_0 > 0
\]

and suppose that \( h(x) = x^{1-\gamma} f(x) \) is a solution of either of the Riccati equations. Then

\[
\mathbb{E}_x[e^{-\lambda X_t}] = U_{\lambda}(x, t)
\]

where the value of \( U_{\lambda}(x, t) \) is given by taking \( u_0 = 1 \)

We have now all the necessary tools to properly a wide class of partial differential equations.
2.2.2 First Riccati equation

When dealing with PDE’s whose drift satisfies the first of the Riccati equation we have in general good analytic results. The main result for this class of processes is the following.

**Theorem 2.2.7.** Suppose \( \gamma \neq 0 \) and \( h(x) = x^{1-\gamma} f(x) \) is a solution of the first Riccati equation

\[
\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma Ax^{2-\gamma} + B
\]

Then the PDE (2.19) has a symmetry solution of the form

\[
U_\epsilon(x, t) = \frac{1}{(1 + 4\epsilon t)^{\frac{2}{2-\gamma}}} \exp \left[ -\frac{4\epsilon(x^{2-\gamma} + 2\sigma(2-\gamma) \beta t)}{\sigma(2-\gamma)^2 (1 + 4\epsilon t)} \right] \times (2.20)
\]

\[
\times \exp \left[ \frac{1}{2\sigma} \left( F \left( \frac{x}{(1 + 4\epsilon t)^{\frac{2}{2-\gamma}}} \right) - F(x) \right) \right] u \left( \frac{x}{(1 + 4\epsilon t)^{\frac{2}{2-\gamma}}}, \frac{t}{1 + 4\epsilon t} \right)
\]

with \( F''(x) = \frac{f(x)}{x^\gamma} \) and \( u \) is a solution of the PDE. Thus for sufficiently small \( \epsilon \) \( U_\epsilon \) is a solution of (2.19) whenever \( u \) is. If \( u(x, t) = u_0(x) \) with \( u_0 \) an analytic, stationary solution then there is a fundamental solution \( p(x, y, t) \) of (2.19) such that

\[
\int_0^\infty e^{\lambda y^{2-\gamma}} u_0 p(x, y, t) dy = U_\lambda(x, t)
\]

where \( U_\lambda(x, t) = U_\lambda(x, t) \).

Further if \( u_0 = 1 \) then \( \int_0^\infty p(x, y, t) dy = 1 \).

**Proof.** Using the Lie algorithm we can see that (2.19) has an infinitesimal symmetry of the form

\[
v = \frac{8xt}{2 - \gamma} \partial_x + 4\epsilon t \partial_t - \left( \frac{4x^{2-\gamma}}{\sigma(2-\gamma)^2} + \frac{4x^{1-\gamma} tf(x)}{\sigma(2-\gamma)^2} + \beta t + 4At^2 \right) u \partial_u
\]

where \( \beta = \frac{4(1-\gamma)}{2-\gamma} \). Exponentiating this symmetry we will get \( U_\epsilon \) in (2.20).

A solution \( U_\lambda(x, t) \) should be obtained as

\[
U_\lambda(x, t) = \int_0^\infty U_\lambda(y, 0) p(x, y, t) dy = \int_0^\infty e^{\lambda y^{2-\gamma}} u_0(y) p(x, y, t) dy
\]

We thus need to show that \( U_\lambda \) is the generalized Laplace transform of some distribution \( u_0 p \). Since

\[
\int_0^\infty e^{\lambda y^{2-\gamma}} u_0(y) p(x, y, t) dy = \int_0^\infty e^{\lambda z} u_0(z^{\frac{1}{2-\gamma}}) p(x, z^{\frac{1}{2-\gamma}}, t) z^{\frac{1}{2-\gamma} - 1} dz
\]

Thus \( U_\lambda(x, t) \) can be written as a product of \( \lambda^\nu \) for some value \( \nu \) and analytic function \( G(1/\lambda) \). Any function which is analytic in \( 1/\lambda \) is a Laplace transform. Further \( \lambda^\nu \) is a Laplace transform. Hence \( U_\lambda(x, t) \) is a Laplace transform.

If we integrate a test function \( \varphi(\lambda) \) with sufficient rapid decay against \( U_\lambda(x, t) \) then \( u(x, 0) = \int_0^\infty \varphi(\lambda) \lambda^{-\nu} \pi^{\nu} |\Gamma(\nu)| \lambda^{\nu + 1/2 - 1/2} d\lambda \).
\[
\int_0^\infty U_\lambda(x,0)\varphi(\lambda)d\lambda \text{ is a solution of (2.19). Further}
\]

\[
u(x,0) = \int_0^\infty U_\lambda(x,0)\varphi(\lambda)d\lambda = \int_0^\infty u_0(x)e^{-\lambda x^2-\gamma}\varphi(\lambda)d\lambda = u_0(x)\varsigma(x)
\]

where \(\varsigma\) is the generalized Laplace transform of \(\varphi\). Now by Fubini’s theorem

\[
\int_0^\infty u_0(y)\varsigma_p(y)p(x,y,t)dy = \int_0^\infty \int_0^\infty u_0(y)\varphi(\lambda)p(x,y,t)e^{-\lambda x^2-\gamma}d\lambda dy =
\]

\[
\int_0^\infty \int_0^\infty u_0(y)\varphi(\lambda)p(x,y,t)e^{-\lambda x^2-\gamma}dyd\lambda = 
\int_0^\infty (y)\varphi(\lambda)U_\lambda(x,t)dx = u(x,t)
\]

But \(u(x,0) = u_0(x)\varsigma_p(x)\). Therefore integrating initial data \(u_0\varsigma_p\) against \(p\), the resulting function solves the Cauchy problem for (2.19). This proves that \(p\) is a fundamental solution.

The previous theorem gives us strong results when a PDE of the form (2.19) with \(g = 0\) is considered. But it is often the case that we are dealing with equations with \(g\) different from 0. We will now state some results for such a more general case. We will treat only the case of some specific function \(g\) that may arise in finance. Results on different functions can be found in [Cra09; CL09; CL07]. Anyway specific case are to be treated independently.

Setting \(g(x) = \mu\) changes some of the results we have previously introduced. First of all we cannot any more take \(u_0 = 1\) as the stationary solution. How do we find then the right stationary solution? Recall that the stationary solution is not unique. Furthermore different stationary solutions will in general lead to different fundamental solutions. We have first of all find a way to choose the right stationary solution \(u_0\) in order to be able to compute the desired expectatin through theorem 2.1.3. We will show how the choice of \(u_0\) is made using an example.

**Example 2.2.1.** Let us consider the process \(X = \{X_t : t \geq 0\}\) where

\[
\begin{aligned}
\begin{cases}
    dX_t = \frac{\sigma}{x}dt + dW_t \\
    X_0 = x
\end{cases}
\end{aligned}
\]

This is better know as Bessel process. We use this process since its transition density is known. Using the usual notation we have \(\sigma = \frac{1}{2}, \gamma = 0\) and \(f(x) = \frac{\sigma}{2}\). Setting thus \(h(x) = a\) we have \(A = 0\) in the Riccati equation. Our aim is to evaluate

\[
\mathbb{E}_x \left[ e^{\frac{\sigma}{2} \int_0^t \frac{1}{x_s} dW_s} \phi(X_t) \right]
\]

Therefore we have to set \(g(x) = \frac{\mu}{4x^2}\) in order to study the PDE

\[
u_t = \frac{1}{2} u_{xx} + \frac{a}{x}u_x - \frac{\mu}{4x^2}u
\]
Finding a fundamental solution of this PDE we will be able to evaluate the desired expectation.

We require a fundamental solution that reduces to the known transition density of the process at \( \mu = 0 \). Computing the stationary solution we find \( u_0 = x^d \) where \( d = \frac{1}{2} - a + \sqrt{\frac{a}{2} + (a - \frac{1}{2})^2} \) to be the right one. The discrimination point is that as \( \mu \to 0 \) \( u_0 \to 1 \). Craddock proved in Craddock and Lennox [CL09] that if \( \mu = 0 \) the transition density in fact reduces to the transition density of the Bessel process.

We have then found a way to choose the right stationary solution. Anyway we have not said anything about how to recover these stationary solutions.

Given the function \( h(x) \), in order to obtain the stationary solution \( u_0 \) we need to solve

\[
\sigma x^\gamma u_{xx} + f(x)u_x - g(x)u = 0
\]

We divide by \( x^{\gamma-1} \) to be able to rewrite the PDE as

\[
\sigma x u_{xx} + h(x)u_x - x^{\gamma-1}g(x)u = 0
\]

If \( u(x) = \tilde{u}(x)e^{-\frac{1}{2}\int \frac{h(s)}{s} ds} \) then \( 2\sigma^2 x^2 \tilde{u}''(x) - (2\sigma Ax^{2-\gamma} + B)\tilde{u}(x) = 0 \). Finally we get through the substitution \( z = x^{2-\gamma} \), \( \tilde{u}(x) = w(x^{2-\gamma}) \) the following ODE

\[
2\sigma^2 (2 - \gamma)^2 z^2 w''(z) + 2\sigma^2 (2 - \gamma) (1 - \gamma) z w'(z) - (2\sigma Az + B)w(z) = 0 \tag{2.21}
\]

This equation has always Bessel functions solutions.

**Lemma 2.2.8.** The PDE

\[
u_t = \sigma xu_{xx} + f(x)u_x - \frac{\mu}{x} u
\]

with \( f \) solution of the Riccati equation

\[
\sigma x f' - sf + \frac{1}{2}f^2 = Ax + B
\]

has a stationary solution \( u_0^\mu(x) \) which has the property that \( u_0^\mu(x) = 1 \)

**Theorem 2.2.9.** Suppose that \( f \) is a solution of

\[
\sigma x f' - sf + \frac{1}{2}f^2 = Ax + B
\]

Then the PDE

\[
u_t = \sigma xu_{xx} + f(x)u_x - \frac{\mu}{x} u, \quad 2B + \sigma^2 + 4\mu\sigma > 0
\]

has a fundamental solution

\[
p(x,y,t) = \frac{\sqrt{x}}{\sigma t} \ e^{-\frac{F(x)}{2\sigma t}} \frac{(x+y)}{\sigma t^2} \frac{\gamma_1}{u_0^\mu(y)} \times
\]

(2.22)
\[ F' = \frac{I(x)}{x} \] and \( \nu = \frac{1}{\sigma} \sqrt{2B + \sigma^2 + 4\sigma \mu} \) satisfies \( |\nu| < 1 \). From which we may calculate

\[ E_x [e^{-\lambda X_t} - \mu \int_0^t ds X_s] = \int_0^\infty e^{-\lambda y} p(x, y, t) dy \]

**Proof.** Let \( f(x) = 2\sigma^2 \frac{d^2}{dy} \) where \( y(x) = \sqrt{x} I_\nu \left( \frac{\sqrt{2Ax}}{\sigma} \right) + c_2 I_{-\nu} \left( \frac{\sqrt{2Ax}}{\sigma} \right) \).

Using the substitution \( u_0 = e^{-\frac{F(x)}{2\sigma^2}} v(x) \) we have that \( v \) has to satisfy

\[ 2\sigma x^2 v''(x) - (Ax + B + 2\sigma \mu) v(x) = 0 \]

which has solution \( \sqrt{x} I_{\pm \nu} \left( \frac{\sqrt{2Ax}}{\sigma} \right) \). Setting now

\[ u_0^\mu(x) = \sqrt{x} e^{-\frac{F(x)}{2\sigma^2}} \left( c_1 I_\nu \left( \frac{\sqrt{2Ax}}{\sigma} \right) + c_2 I_{-\nu} \left( \frac{\sqrt{2Ax}}{\sigma} \right) \right) \]

We then have \( u_0^\mu(y) = 1 \). Now

\[ \int_0^\infty e^{-\lambda y} u_0^\mu(y) p(x, y, t) dy = \]

\[ = \sqrt{x} e^{-\frac{F(x)}{2\sigma^2}} \frac{\frac{2Ax}{\sigma} + \frac{4\sigma^2}{1 + \lambda \sigma t}}{1 + \lambda \sigma} \times \]

\[ \times \left( c_1 I_\nu \left( \frac{\sqrt{2Ax}}{\sigma (1 + \lambda \sigma t)} \right) + c_2 I_{-\nu} \left( \frac{\sqrt{2Ax}}{\sigma (1 + \lambda \sigma t)} \right) \right) \]

Using now the fact that

\[ \mathcal{L}^{-1} \left[ \frac{1}{\lambda} \exp \left( \frac{m^2 + n^2}{\lambda} \right) I_d \left( \frac{2mn}{\lambda} \right) \right] = I_d(2m \sqrt{y}) I_d(2n \sqrt{y}), \quad d > -1 \]

Inverting the Laplace transform we recover the fundamental solution. The expectation is found via the Feynman-Kac theorem 2.1.3.

### 2.2.3 Second Riccati equation

As said the case of the first Riccati equation is in general analytically good. More complicated is the case of the second Riccati equation. The structure is as previously. We will first state a results for the case \( g = 0 \). Then we will consider some particular function \( g \).

**Theorem 2.2.10.** Let us consider the PDE

\[ u_t = \sigma x^2 u_{xx} + f(x) u_x - g(x) u \]  

(2.23)
and suppose that \(g\) and \(h(x) := x^{1-\gamma} f(x)\) satisfy the second Riccati equation

\[
\sigma x f' - \sigma f + \frac{1}{2} f^2 = -\frac{A}{2(2-\gamma)} x^{4-2\gamma} + \frac{B}{2-\gamma} x^{2-\gamma} + C
\]  

(2.24)

Let \(u_0\) be a stationary, analytic solution of (??). Then (??) has a symmetric solution of the form

\[
\bar{U}_\gamma(x,t) = (1 + \epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon \sinh(\sqrt{A}))^{-c} \times
\]

\[
\times \left[ \cosh\left(\frac{\sqrt{A}t}{2}\right) + (1 + 2\epsilon) \sinh\left(\frac{\sqrt{A}t}{2}\right) \right] \left[ \cosh\left(\frac{\sqrt{A}t}{2}\right) - (1 - 2\epsilon) \sinh\left(\frac{\sqrt{A}t}{2}\right) \right]^{-\frac{B}{2\sigma(2-\gamma)}} \exp \left( \frac{1}{2\sigma} F(x) - \frac{Bt}{2\sigma(2-\gamma)} \right) \times
\]

\[
\times \exp \left( -\sqrt{A} \epsilon x^{2-\gamma}(\cosh(\sqrt{A}t) + \epsilon \sinh(\sqrt{A})) \right) \left( \frac{x}{(1 + \epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon \sinh(\sqrt{A}))^{\frac{1}{2-\gamma}}} \right) \times
\]

\[
\times u_0 \left( \frac{x}{(1 + \epsilon^2(\cosh(\sqrt{A}t) - 1) + 2\epsilon \sinh(\sqrt{A}))^{\frac{1}{2-\gamma}}} \right)
\]

with \(F'(x) = \frac{f'(x)}{x}\) and \(c = \frac{1-\gamma}{2-\gamma}.

Furthermore it exists a fundamental solution \(p(x, y, t)\) of (??):

\[
\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0 p(x, y, t) dy = U_\lambda(x, t)
\]

where \(U_\lambda(x, t) = \bar{U}_\gamma \frac{x}{\gamma} (x, t)\).

**Proof.** It can be shown that if \(f\) satisfies the condition than a basis for the symmetric group is spanned by

\[
\begin{align*}
\mathbf{v}_1 &= \frac{\sqrt{A}}{2-\gamma} \cosh(\sqrt{A}t) \partial_x + e^{\sqrt{A}t} \partial_t - \left( \frac{A x^{1-\gamma}}{2\sigma(2-\gamma)^2} + \frac{\sqrt{A} x^{1-\gamma}}{2\sigma(2-\gamma)^3} e^{\sqrt{A}t} u \partial_u - \alpha u e^{\sqrt{A}t} \partial_u \right) \\
\mathbf{v}_2 &= \frac{\sqrt{A}}{2-\gamma} - \sqrt{A} t \partial_x + e^{-\sqrt{A}t} \partial_t - \left( \frac{A x^{1-\gamma}}{2\sigma(2-\gamma)^2} - \frac{\sqrt{A} x^{1-\gamma}}{2\sigma(2-\gamma)^3} e^{-\sqrt{A}t} u \partial_u \right) \\
\mathbf{v}_3 &= \partial_t \\
\mathbf{v}_4 &= u \partial_u
\end{align*}
\]

where we have denoted by \(\alpha = \frac{1-\gamma}{2(2-\gamma)} \sqrt{A} + \frac{B}{2\sigma(2-\gamma)}\) and \(\beta = -\frac{(1-\gamma)}{2(2-\gamma)} \sqrt{A} + \frac{B}{2\sigma(2-\gamma)}\).

The key is to compose the basis such that when setting \(t = 0\) the symmetric group reduces to a generalized Laplace transform. This happens to be

\[
\mathbf{v} = \frac{2\sqrt{A} x}{2-\gamma} \sinh(\sqrt{A}t) \partial_x + 2(\cosh(\sqrt{A}t) - 1) \partial_t - \frac{1}{\sigma} g(x, t) u \partial_u
\]
with
\[ g(x, t) = \left( \frac{Ax^{2-\gamma}}{(2-\gamma)^2} + \frac{B}{2-\gamma} \right) \cosh(\sqrt{At}) + \sqrt{A} \sinh(\sqrt{At}) \left( \frac{x^{1-\gamma}}{2-\gamma} f(x) + \sigma \epsilon \right) \]

Exponentiating as explained before gives the symmetric group \( U_\epsilon(x, t) \) in (2.25).

The change of parameter \( \epsilon \mapsto \sigma (2-\gamma)^2 \lambda \sqrt{A} \) implies \( U_\lambda(x, 0) = e^{-\lambda x^{2-\gamma}} \) is the wanted generalized Laplace transform.

Now if \( p(x, y, t) \) is a fundamental solution of (2.23) then
\[
\int_0^\infty = e^{-\lambda x^{2-\gamma}} u_0(y)p(x, y, t)dy = U_\lambda(x, t)
\]
(2.26)

This anyway has to be proven.

Since the drift is an analytic function then \( U_\lambda \) can be written as a product of functions analytic in \( \frac{1}{\lambda} \) and any function analytic in \( \frac{1}{\lambda} \) is a Laplace transform.

Let us now suppose (2.26) holds for some distribution \( p \). We have to prove that \( p \) is a fundamental solution for the PDE. The fact that \( U_\lambda(x, t) \) together with (2.26) implies \( p \) is a solution for any fixed \( y \). Integrating now a test function \( \varphi(\lambda) \) with sufficiently rapid decay against \( U_\lambda \). Then the function \( u(x, t) = \int_0^\infty U_\lambda(x, t) \varphi(\lambda) d\lambda \) is a solution of (2.2) from lemma 2.1.2.

Furthermore we have
\[
u(x, 0) = \int_0^\infty U_\lambda(x, 0) \varphi(\lambda) d\lambda = \int_0^\infty u_0(x) e^{-\lambda x^{2-\gamma}} \varphi(\lambda) d\lambda = u_0(x) \varsigma_\gamma(x)
\]
where \( \varsigma_\gamma \) is the generalized Laplace transform of \( \varphi \). Further by Fubini’s theorem
\[
\int_0^\infty u_0(y) \varsigma_\gamma(y)p(x, y, t)dy = \int_0^\infty \int_0^\infty u_0(y) \varphi(\lambda)p(x, y, t)e^{-\lambda x^{2-\gamma}} d\lambda dy \varphi(\lambda) d\lambda =
\int_0^\infty \int_0^\infty u_0(y) \varphi(\lambda)p(x, y, t)e^{-\lambda x^{2-\gamma}} dy d\lambda \varphi(\lambda) d\lambda =
\int_0^\infty (y) \varphi(\lambda)U_\lambda(x, t)dx = u(x, t)
\]

But \( u(x, 0) = u_0(x) \varsigma_\gamma(x) \). Therefore integrating initial data \( u_0 \varsigma_\gamma \) against \( p \), the resulting function solves the Cauchy problem for (2.23). This proves that \( p \) is the fundamental solution.

Again for the particular case of \( g = 0 \) a strong result is stated in theorem (2.2.10). What can be said in stead for the general case of a non null function \( g \)?

The problem when dealing with a solution of the second Riccati equation is that the criterion for searching fundamental solutions we have introduced before in lemma 2.2.8 does not apply here. In fact it is not necessary that a stationary solution does not reduce to 1 when setting \( \mu = 0 \). We have thus to find a different approach from the inversion via Laplace transform.

An alternative approach to the construction of the necessary fundamental solution is to use group invariant solutions. We mean by group invariant solution is a solution which is left invariant under the
Then \( \exists \) with \( \alpha \)

Theorem 2.2.11. Let us suppose \( \beta \) where \( \nu \)

same time we require the boundary \( t \)

therefore look for symmetries of the form \( v \)

is given by \( U \)

equations we always have a solution of the form

\[
\Gamma(\alpha, b, z) = \frac{\Gamma(b-1)}{\Gamma(a)} \frac{\Gamma(1-b)}{\Gamma(a+b+1)} F_1(a-b+1, 2-b, z)
\]

Anyway we can say that when dealing with an equation of the second family of the Riccati equations we always have a solution of the form

\[
w(z) = x^\beta e^{-\sqrt{A}x} \left( c_{11} F_1(\alpha, 2\beta, \sqrt{A}x) + c_2 U(\alpha, 2\beta, \sqrt{A}x) \right)
\]

where \( \alpha = \frac{b + \sigma + \sqrt{\frac{b^2}{2} + 1}}{2\sigma}, \beta = \frac{1}{2} \left( \sqrt{\frac{b^2}{2} + 1} + 1 \right) \), \( F_1(a, b, z) = M(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{n+1} n!} \)

with \( a^{(n)} = a(a + 1) \ldots (a + n - 1) \) is the Kummer’s confluent hypergeometric function and \( U(a, b, z) = \frac{\Gamma(b-1)}{\Gamma(a)} \frac{\Gamma(1-b)}{\Gamma(a+b+1)} F_1(a-b+1, 2-b, z) \) is the Tricomi’s confluent hypergeometric function.

Theorem 2.2.11. Let us suppose \( f \) is a solution of the second Riccati equation

\[
\sigma f + \frac{1}{2} f^2 + 2\sigma \mu x^2 = \frac{1}{2} Ax^2 + Bx + C, \quad \text{with} \quad A > 0
\]

Then \( \exists \) a fundamental solution of

\[
u_t = \sigma x u_x + f(x) u_x - \mu x u, \quad x \geq 0
\]

of the form

\[
p(x, y, t) = \frac{\sqrt{Ax} ye^{-(F(x)-F(y)/(2\sigma))}}{2\sigma \sinh(\sqrt{At}/2)} \exp \left( -\frac{Bt}{2\sigma} - \frac{\sqrt{A}(x+y)}{2\sigma \tanh(\sqrt{At}/2)} \right)
\]

(2.27)

(2.29)

where \( \nu = \frac{\sigma x + \sqrt{A}C}{\sigma} \) and \( F'(x) = \frac{f(x)}{x} \).

Proof. Whenever the drift \( f \) satisfies the given Riccati equation, then a basis for the symmetric group is given by

\[
\begin{align*}
v_1 &= xe^{\sqrt{A}t} \partial_x + e^{\sqrt{A}t} \frac{e^{\sqrt{A}t}}{\sqrt{A}} \partial_t - \frac{1}{2\sigma} \left( \sqrt{A}x + f(x) + \frac{B}{\sqrt{A}} \right) e^{\sqrt{A}t} u \partial_u \\
v_2 &= -xe^{-\sqrt{A}t} \partial_x + e^{-\sqrt{A}t} \frac{e^{\sqrt{A}t}}{\sqrt{A}} \partial_t - \frac{1}{2\sigma} \left( \sqrt{A}x - f(x) + \frac{B}{\sqrt{A}} \right) e^{-\sqrt{A}t} u \partial_u \\
v_3 &= \partial_t \\
v_4 &= u \partial_u \\
v_5 &= \beta(x, t) \partial_u
\end{align*}
\]

\( \beta \) is an arbitrary solution of the PDE. Obviously adding this will not give us an invariant solution. We therefore look for symmetries of the form \( v = \sum_{i=1}^{4} c_i v_i \) which preserves the boundary conditions.

We require \( x = 0 \) to be preserved by the action of \( v \). That is \( v(x) = 0 \) when \( x = 0 \). At the same time we require the boundary \( t = 0 \) to be preserved. As before this is \( v(t) = 0 \) when \( t = 0 \).
The last boundary condition $u(x, 0) = \delta(x - y)$ must be preserved that is again $v(x - y) = 0$ when $u(x, 0) = \delta(x - y)$. In order this conditions to be satisfied we need $c_1 = c_2$, $c_3 = -\frac{2}{\sqrt{A}}c_1$ and $c_4 = 2\frac{(Ay + B)}{\sigma\sqrt{A}}c_1$.

Therefore the action we are looking for will be of the form

$$v = c_3 \left( -\frac{2}{\sqrt{A}} v_3 + v_1 + v_2 + 2\frac{(Ay + B)}{\sigma\sqrt{A}} v_4 \right)$$

Invariants of the action, as explained in the first part, are found solving $v(\eta) = 0$. They can be found through the method of characteristic. We can thus consider $\eta$ and $v$ where

$$\eta = \frac{x}{4 \sinh^2 \left( \frac{\sqrt{A}t}{2} \right)}$$

$$u = \exp \left( -\frac{Bt + F(x) - F(y)}{2\sigma} - \frac{\sqrt{A}(x + y)}{2\sigma \tanh(\sqrt{A}t/2)} \right) v \left( \frac{x}{4 \sinh^2 \left( \frac{\sqrt{A}t}{2} \right)} \right)$$

Now $v(\eta)$ must satisfy $2\sigma^2 \eta^2 v''(\eta) - (C + 2Ay\eta)v(\eta) = 0$ when $u$ is a solution of the PDE. Hence

$$v(\eta) = \sqrt{\eta} \left( C_1(\eta) I_{\nu} \left( \frac{2\sqrt{A}y\eta}{\sigma} \right) + C_2(\eta) I_{-\nu} \left( \frac{2\sqrt{A}y\eta}{\sigma} \right) \right)$$

since the solution is a group invariant solution for the action generated by $v$ the result follows from substituting $v$ and $\eta$ into (2.31).
Chapter 3

Financial markets

3.1 The Cox-Ingersoll-Ross (CIR) model

The CIR model is one of the most used process in finance. It is usually used to describe the behaviour of interest rate. It has several properties that make such a process the ideal SDE when dealing with financial markets. For an extensive treatment of such a model we refer to Bri;Cox.

3.1.1 CIR transition density function

We skip any financial topic and we will focus on the CIR process and on its symmetric group.

We will study the stochastic process \( X = \{X_\tau : \tau \geq 0\} \) solution of the CIR model, see CIR for details, described by the following SDE

\[
\begin{align*}
\frac{dX_\tau}{X_\tau} &= k(\theta - X_\tau)d\tau + \sigma \sqrt{X_\tau}dW_\tau \\
X_0 &= x 
\end{align*}
\]  

(3.1)

We want to recover the fundamental solution associated to the SDE (3.1) exploiting the method described previously.

In particular the PDE associated to equation (3.1) is of the form described by equation (2.23), i.e. it reads as follows

\[
u_{\tau} = \frac{\sigma^2}{2} xu_{xx} + k(\theta - x)u_x .
\]

(3.2)

According to notation introduced in Th. (2.2.10), we have

\[
\gamma = 1 ; g(x) = 0 ; f(x) = k(\theta - x) ; h(x) = f(x) = k(\theta - x) ,
\]

and in order \( h(x) \) to satisfy an equation of type (2.24) we obtain

\[
-\frac{\sigma^2}{2}xk + \frac{\sigma^2}{2}xk - \sigma k\theta + \frac{(k\theta)^2}{2} + \frac{k^2x^2}{2} - k^2\theta x = A\frac{x^2}{2} + Bx + C ,
\]

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We can finally apply (2.2.10) with \( F(x) = k\theta \ln(x) - kx \) and \( u_0(x) = 1 \) to obtain the desired solution, namely

\[
U_{\frac{x^2}{2} \frac{\tau^2}{2}}(x, \tau) = \frac{2k^2 \theta e^{2k^2 \tau \frac{x^2}{2}}}{(\frac{\sigma^2}{2} \lambda(e^{k\tau} - 1) + k e^{k\tau})} \exp \left( \frac{-\lambda kx}{(\frac{\sigma^2}{2} \lambda(e^{k\tau} - 1) + k e^{k\tau})} \right),
\]

in fact we have that

\[
U_e(x, \tau) = \frac{\cosh(\frac{kx}{\sigma}) + (1 + 2\epsilon) \sinh(\frac{kx}{\sigma})}{\cosh(\frac{kx}{\sigma}) - (1 - 2\epsilon) \sinh(\frac{kx}{\sigma})} \frac{e^{\frac{k\theta}{\sigma^2}}}{e^{\frac{k\theta}{\sigma^2}} \sinh(\frac{k\theta}{\sigma^2})} \exp \left( \frac{-\lambda kx}{(\frac{\sigma^2}{2} \lambda(e^{k\tau} - 1) + k e^{k\tau})} \right) \times
\]

\[
\times \exp \left( \frac{-kx(\cosh(k\tau) + \epsilon \sinh(k\tau))}{\sigma^2(1 + 2\epsilon)(\cosh(k\tau) - 1) + 2\epsilon \sinh(k\tau))} \right) \times \exp \left( \frac{k\theta}{\sigma^2} \log \left( \frac{x(1 + e^2(\cosh(k\tau) - 1) + 2\epsilon \sinh(k\tau))}{x} \right) \right) \times \exp \left( \frac{kx}{\sigma^2} \frac{1 - 2\epsilon(\cosh(k\tau) - \epsilon \sinh(k\tau))}{1 + 2\epsilon(\cosh(k\tau) - 1) + 2\epsilon \sinh(k\tau))} \right).
\]

Let us focus our attention separately on Part1 and Part2 of the equation (3.3). For what concerns Part1 we have

\[
\text{Part1} = \left[ \frac{\cosh(\frac{kx}{\sigma}) + (1 + 2\epsilon) \sinh(\frac{kx}{\sigma})}{\cosh(\frac{kx}{\sigma}) - (1 - 2\epsilon) \sinh(\frac{kx}{\sigma})} \right] \frac{e^{\frac{k\theta}{\sigma^2}}}{e^{\frac{k\theta}{\sigma^2}} \sinh(\frac{k\theta}{\sigma^2})} \exp \left( \frac{-\lambda kx}{(\frac{\sigma^2}{2} \lambda(e^{k\tau} - 1) + k e^{k\tau})} \right) \times
\]

\[
\times \left( \frac{e^{k\tau} + e^{-k\tau} + (1 + 2\epsilon) (e^{k\tau} - e^{-k\tau})}{e^{k\tau} + e^{-k\tau} - (1 - 2\epsilon) (e^{k\tau} - e^{-k\tau})} \right) \exp \left( \frac{k\theta}{\sigma^2} \log \left( \frac{1 + e^2(\cosh(k\tau) - 2) + \epsilon(\cosh(k\tau) - 2) + \epsilon(\cosh(k\tau) - 2 \epsilon)}{1 - e^{2k\tau} - 2 \epsilon} \right) \right)
\]

\[
= \left( \frac{e^{k\tau} + e^{-k\tau} + (1 + 2\epsilon) (e^{k\tau} - e^{-k\tau})}{e^{k\tau} + e^{-k\tau} - (1 - 2\epsilon) (e^{k\tau} - e^{-k\tau})} \right) \exp \left( \frac{k\theta}{\sigma^2} \log \left( \frac{1}{e^{2k\tau} + e^{-2k\tau} - 2 \epsilon^2 + 2} \right) \right)
\]

\[
= \left( \frac{e^{k\tau}}{(e^{-k\tau} (1 + \epsilon)^2)} \right) \frac{k\theta}{\sigma^2} \exp \left( \frac{k\theta}{\sigma^2} \log \left( \frac{1}{e^{2k\tau} + e^{-2k\tau} - 2 \epsilon^2 + 2} \right) \right)
\]

hence, taking \( \epsilon = \frac{\sigma^2 \lambda}{2k} \), we get

\[
\text{Part1} = \left( \frac{2k e^{k\tau}}{\sigma^2 \lambda (e^{k\tau} - 1) + 2k e^{k\tau}} \right)^{\frac{k\theta}{\sigma^2}}
\]

Concerning the second part of equation (3.3), we have
Part 2 = \exp \left( \frac{kx}{\sigma^2} \left( \frac{1 + 2c^2(\cosh(k\tau) - 1) + 2c \sinh(k\tau) - 1 - 2c(\cosh(k\tau) - 2c^2 \sinh(k\tau))}{1 + 2c^2(\cosh(k\tau) - 1) + 2c \sinh(k\tau)} \right) \right) = \\
= \exp \left( \frac{kx}{\sigma^2} \left( \frac{2c^2 (\cosh(k\tau) - 1) + 2c \sinh(k\tau) - 2c \cosh(k\tau) - 2c^2 \sinh(k\tau)}{1 + 2c^2(\cosh(k\tau) - 1) + 2c \sinh(k\tau)} \right) \right) = \\
= \exp \left( \frac{kx}{\sigma^2} \left( \frac{2c^2 \left( \frac{e^{c\tau + e^{-c\tau}}}{2} - 1 \right) + 2c \left( \frac{e^{c\tau} - e^{-c\tau}}{2} \right) - 2c \left( \frac{e^{c\tau + e^{-c\tau}}}{2} \right)}{1 + 2c^2(\cosh(k\tau) - 1) + 2c \sinh(k\tau)} \right) \right) = \\
= \exp \left( \frac{kx}{\sigma^2} \left( \frac{e^{c\tau}(e^2 - e^2 + e - e)}{1 + 2c^2 \left( \frac{e^{c\tau} + e^{-c\tau}}{2} - 1 \right) + 2c \left( \frac{e^{c\tau} - e^{-c\tau}}{2} \right)} \right) \right) = \\
= \exp \left( \frac{kx}{\sigma^2} \left( \frac{e^{-c\tau}(2e^2 - 2c^2) - 2e^2 \left( \frac{e^{c\tau} + e^{-c\tau}}{2} - 1 \right) + 2c \left( \frac{e^{c\tau} - e^{-c\tau}}{2} \right)}{1 + 2c^2(\cosh(k\tau) - 1) + 2c \sinh(k\tau)} \right) \right) = \\
= \exp \left( \frac{-2kx\epsilon}{\sigma^2 \epsilon^2 + e^{k\tau} \sigma^2(1 + \epsilon)} \right),

thus, taking \( \epsilon = \frac{2c^2}{k} \), we obtain \( Part 2 = \exp \left( \frac{-2kx\lambda}{\sigma^2 \lambda(e^{k\tau} - 1) + 2ke^{k\tau}} \right). \)

Combining \( Part 1 \) and \( Part 2 \), we finally have

\[
U_2\left(\frac{x^2 + y^2}{2\sigma^2}\right)(x, \tau) = \int_0^\infty e^{-\lambda y} p(x, y, \tau) dy = \\
= \frac{2k^2}{\sigma^2 e^{k\tau}} \frac{2k^2}{\sigma^2 e^{k\tau}} \exp \left( \frac{-2\lambda kx}{\sigma^2 \lambda(e^{k\tau} - 1) + 2ke^{k\tau}} \right),
\]

which is nothing but the Laplace transform of \( p(x, y, \tau) \), moreover it can be inverted, see Craddock [Cra09] and Craddock and Lennox [CL09] for details, to have that the fundamental solution of (3.2) reads as follows

\[
\rho_{CIR}(x, y, \tau) = \frac{2ke^{k(\frac{x^2}{\sigma^2} + 1)\tau}}{\sigma^2(e^{k\tau} - 1)} \left( \frac{y}{x} \right)^{\frac{1}{2}} \exp \left( \frac{-2k(x + e^{k\tau} y)}{\sigma^2(e^{k\tau} - 1)} \right) I_{\nu} \left( \frac{2k\sqrt{xy}}{\sigma^2 \sinh(\frac{\tau}{2})} \right), \quad (3.4)
\]

where \( \nu := \frac{2k\theta}{\sigma^2} - 1 \). We have found a fundamental solution but we are not done yet, since this does not guarantee us that this is a transition density function. Not even the condition, which is in fact satisfied, \( \int_0^\infty p(x, y, \tau) dy = 1 \), is sufficient to this intent. Using now proposition 2.2.5 we can conclude that (3.28) is in fact the transition density function of the CIR process.

Moreover, exploiting theorem 2.2.11, we can compute the ZCB fair price, see, e.g., Brigo and Mercurio [BM06] for a description of such type of bond. In fact, supposing the ZCB associated interested rate is driven by the CIR process defined by equation (3.1), then its fair price is given by theorem (2.2.11) setting

\[
\mu = 1 ; \quad A = k^2 + 2\sigma^2 ; \quad B = -k^2 \theta ; \quad C = \frac{\theta k}{2} (\theta k - \sigma^2) ; \\
F(x) = \theta k \ln(x) - kx ; \quad C_1 = 1 ; \quad C_2 = 0
\]

We recall the definition of fundamental solution for a parabolic operator. Given a differential
operator $u_\tau - Lu = 0$ with initial datum $u(0, x) = f(x)$ for $x \in \Omega$ and $\tau \in [0, T]$, a fundamental solution is a kernel $p(x, y, \tau)$ such that: (i) for fixed $y$, $p(x, y, \tau)$ is a solution of the PDE on $\Omega \times (0, T]$; (ii) $u(x, \tau) = \int_{\Omega} f(x)p(x, y, \tau)dy$ is a solution of the Cauchy problem for a given initial datum $f$.

Therefore we can recover a classical solution from the fundamental solution using point (ii) of the previous definition, being $P_t$ the price at time $t$ of the ZCB. Setting

$$P_t = u(x, t) = \int_0^\infty \gamma \exp\left(\frac{\gamma^2(t^2-2\theta k)/2e^{\gamma(t^2-2\theta k)/2}}{\sigma^2}\right) \times \exp\left(\frac{k^2\theta(T-t)}{\sigma^2} - \frac{\gamma(x+y)}{\sigma^2}\right) I_{\sigma^2-2\theta k/\sigma^2} \left(\frac{2\gamma \sqrt{\pi y}}{\sigma^2 \sinh(\gamma(T-t)/2)}\right) dy$$

(3.5)

We can thus recover the following

$$u(x, t) = \exp\left[\frac{-2x \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + k)(e^{\gamma(T-t)} - 1) + 2\gamma} \right] \left[\frac{2\gamma \exp[(\gamma + k)(T-t)/2]}{(\gamma + k)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]$$

with $\gamma = \sqrt{k^2 + 2\sigma^2}$. As already said at the beginning here we have called the time variable $\tau = T - t$ the time left to maturity merely for financial reason.

### 3.1.2 Zero-Coupon Bond Price

The Zero-Coupon bond price plays a fundamental role in financial mathematics. In fact having an analytic formula for the fair price of a ZCB, we can recover the yield curve. Such a curve is of paramount importance in pricing many financial instruments. Again we refer to Brigo and Mercurio [BM06] for an extensive treatment of the topic. We will now evaluate the Zero-Coupon bond price with short rate modelled by the CIR process using Lie’s symmetries.

Let us therefore suppose that the interest rate $R_t$ is governed by the SDE

$$dX_t = k(\theta - X_t)dt + \sigma \sqrt{X_t}dW_t$$

(3.6)

We can say that the zero-coupon price with maturity $T$, recalling that the price at maturity is $P(T, T) = 1$, has fair price given by

$$\mathbb{E}\left[\phi(X_T) \exp(-\int_0^T X_t dt)\right] = \mathbb{E}\left[1 \exp(-\int_0^T X_t dt)\right] = P(0, T)$$

Via Feynman-Kac formula we have that it is solution of the parabolic problem

$$\begin{cases}
  u_t + \frac{\sigma^2}{2} Xu_{xx} + k(\theta - x)u_x - xu = 0 \\
  u(x, T) = 1
\end{cases}$$

(3.7)
The Cox-Ingersoll-Ross (CIR) model

The standard approach is to guess a solution of the form

\[ u(x, t) = a(t, T) \exp[-xb(t, T)] \]

in order to prove that the analytic solution of the PDE (3.7) is of the form

\[ u(x, t) = \exp\left[\frac{-2x}{(\gamma + k)(e^{\gamma(T-t)} - 1)} + 2\gamma\right] \frac{2\gamma \exp [(\gamma + k)(T - t)/2]}{(\gamma + k)(e^{\gamma(T-t)} - 1) + 2\gamma} (3.8) \]

We will use a different approach. Using Lie's algebra we will find equation (3.8) as invariant solution. What we will do in particular to determine a sub-algebra \( F \) of the Lie-algebra admitted by the CIR equation that leaves the final condition and the boundary surfaces unchanged.

Proceeding as usual we find 7 determining equations given the infinitesimal generator of the form

\[ v = \xi(x, t) \partial_x + \tau(x, t) \partial_t + \phi(x, u) \partial_u \]

\[ \xi_u = 0 \quad (3.9a) \]
\[ \tau_u = 0 \quad (3.9b) \]
\[ \phi_{uu} = 0 \quad (3.9c) \]
\[ \tau_x = 0 \quad (3.9d) \]
\[ -2x\sigma^2\phi_{x,u} + \xi_t - 2kx\xi_x + 2\theta k\xi_x + x\sigma^2\xi_{x,x} + 2kx\tau_t - 2\theta k\tau_t + 2k\xi = 0 \quad (3.9e) \]
\[ -2\phi_t - 2x\phi_{u} + 2kx\phi_x - 2k\theta\phi_{x,x} + 2x\tau_t + 2x\phi + 2u\xi = 0 \quad (3.9f) \]
\[ -2x\xi_x + x\tau_t + \xi = 0 \quad (3.9g) \]
\[ -2x\xi_x + x\tau_t + \xi = 0 \quad (3.9h) \]

Solving the previous system we obtain the coefficients (3.33)

\[ \left\{ \begin{align*}
\xi(x, t) &= e^{-\gamma t}(e^{2\gamma t}k_1 + k_2)x \\
\tau(x, t) &= \frac{e^{2\gamma t}k_1 - e^{-\gamma t}k_2 + 2k\gamma}{2} \\
\phi(x, t) &= \frac{e^{-\gamma t}}{\sigma^2}(k_4 e^{\gamma \sigma^2 t} + k_2 (\theta k - \gamma) + x(-\gamma + k\gamma)) + e^{2\gamma t}k_1(-\theta \gamma (k + \gamma) + x(\gamma^2 + k\gamma)) + e^{\gamma t} \sigma \gamma \alpha)
\end{align*} \right. \]

(3.10)

where \( \xi, \tau \) and \( \phi \) are the coefficients of the general infinitesimal generator

\[ v = \xi(x, t) \partial_x + \tau(x, t) \partial_t + u\phi(x, t) \partial_u \]

and \( \alpha(x, t) \) a general solution of equation (3.33).
The system (3.35) allows us to determine a basis for the Lie algebra to be

$$\begin{align*}
v_1 &= e^{\gamma t} \left( x\partial_x + \frac{1}{\gamma} \partial_t + \left( \frac{\theta k}{\sigma^2} \frac{g_k}{\sigma^2} - \frac{g_{k+1}}{\sigma^2} \right) u \partial_u \right) \\
v_2 &= e^{-\gamma t} \left( x\partial_x - \frac{1}{\gamma} \partial_t + \left( \frac{\theta k}{\sigma^2} \frac{g_{k-1}}{\sigma^2} + \frac{g_{k-2}}{\sigma^2} \right) u \partial_u \right) \\
v_3 &= \partial_t \\
v_4 &= u \partial_u
\end{align*}$$

(3.11)

with $\gamma = \sqrt{k^2 + \sigma^2}$.

The commutator table is thus

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>$\gamma v_3$</td>
<td>$-\gamma v_4$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$-\gamma v_3$</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{\gamma} v_1 - \frac{2\theta k^2}{\sigma^2} v_2$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$\gamma v_4$</td>
<td>0</td>
<td>$-\frac{2}{\gamma} v_1 + \frac{2\theta k^2}{\sigma^2} v_2$</td>
<td>0</td>
</tr>
</tbody>
</table>

We are looking for a subalgebra $F$ that leaves invariant the boundary condition such as the boundary surfaces. Therefore we want an *infinitesimal generator* of the form

$$v = \sum_{i=1}^{4} c_i v_i$$

such that

$$v[t - T]|_{t=T} = 0, \quad v[u - 1]|_{u=1} = 0$$

In this case $F$ is spanned by

$$v = \left( k[1 - \cosh(\gamma(t - T))] + \gamma \sinh(\gamma(t - T)) \right) \partial_t + \\
- \gamma \left( k \sinh(\gamma(t - T)) - \gamma \cosh(\gamma(t - T)) \right) x \partial_x + \\
+ 2 \left( k \theta [1 - \cosh(\gamma(t - T))] + \gamma x \sinh(\gamma(t - T)) \right) u \partial_u$$

(3.12)

We have thus to look for an invariant solution of (3.7) that arise from (3.12).

Using the method of characteristic we have

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}$$

(3.13)

where as usual we have denoted by $\xi, \tau, \phi$ the coefficients of $\partial_x, \partial_t$ and $\partial_u$ in (3.12).

Integrating the first and second member in (3.13) gives one invariant

$$X_1 = \left( \gamma (e^{2\gamma (T-t)} - 1) + k(e^{\gamma (T-t)} - 1)^2 \right) \frac{e^{\gamma t}}{x}$$

(3.14)
3.1 The Cox-Ingersoll-Ross (CIR) model

Integrating the second and third member in (3.13) gives instead

$$X_2 = e^{h(x,t)} \left( (\gamma + k)(e^{2\gamma(T-t)} - 1) + 2\gamma \right) \frac{2\theta k}{\sigma^2}$$

(3.15)

where

$$h(x,t) = \frac{2(e^{2\gamma(T-t)} + 1)x}{\gamma(e^{2\gamma(T-t)} - 1) + k(e^{\gamma(T-t)} - 1)^2} + \frac{\theta k(\gamma + k)t}{\sigma^2}$$

Now we have that for an arbitrary function $\Psi$, $X_2 = \Psi(X_1)$ is the most general invariant of (3.12). Therefore the corresponding group-invariant solution must have the form

$$u(x,t) = e^{-h(x,t)} \left( (\gamma + k)(e^{2\gamma(T-t)} - 1) + 2\gamma \right) \frac{2\theta k}{\sigma^2} \times \Psi \left( \left( (\gamma + k) \left( e^{2\gamma(T-t)} - 1 \right) + k(e^{\gamma(T-t)} - 1)^2 \right) \frac{e^{tT}}{x} \right)$$

(3.16)

Substituting now (3.16) into (3.7) leads to

$$y^4 \Psi''(y) - (A y^2 + B y^3) \Psi'(y) - (C + D y) \Psi(y) = 0$$

(3.17)

with

$$A = \frac{4k^2 e^{\gamma T}}{\sigma^2}, \quad B = \frac{2(\theta k - \sigma^2)}{\sigma^2}, \quad C = \frac{16 e^{2\gamma T}(k^2 + \sigma^2)}{\sigma^2}, \quad D = \frac{8\theta k e^{\gamma T}}{\sigma^2}$$

We can now make the change of variable

$$y = \frac{1}{x}, \quad \Psi(y) = v(x)$$

followed by

$$x = z \frac{\sigma^2}{4k^2 e^{\gamma T}}, \quad v(x) = w(z)$$

ending up with the equation

$$zw'' + \left( \frac{2\theta k}{\sigma^2} + z \right) w' - \left( \frac{\sigma^2(k^2 + \sigma^2)}{k^4} + \frac{2\theta}{k} \right) w = 0$$

(3.18)

We can now group the terms depending on $z$ in order to get

$$\left[ z \left( \left( \frac{d}{dz} + \frac{\sigma^2 + k^2}{k^2} \right) \left( \frac{d}{dz} - \frac{\sigma^2}{k^2} \right) \right) + \frac{2\theta k}{\sigma^2} \left( \frac{d}{dz} - \frac{\sigma^2}{k^2} \right) \right] w = 0$$

(3.19)

Introducing now the variable

$$\zeta = \left( \frac{d}{dz} - \frac{\sigma^2}{k^2} \right) w$$

(3.20)

we reduce equation (3.19) into

$$\zeta' + \left( \frac{\sigma^2 + k^2}{k^2} + \frac{2\theta k}{\sigma^2 z} \right) \zeta = 0$$

(3.21)
whose solution is
\[ \zeta = Cz^{-\frac{2\theta k}{\sigma^2}} \exp \left[ - \left( 1 + \frac{\sigma^2}{k^2} \right) z \right] \] (3.22)
with \( C \) a constant of integration. Combining now (3.20) and (3.22) we obtain the differential equation
\[ w' - \frac{\sigma^2}{k^2} w = \zeta(z) \] (3.23)

Now the quadrature is easily done, and setting \( C = 0 \) for boundary condition we obtain
\[ \Psi(y) = K \exp \left[ \frac{4e^{\gamma T}}{y} \right] \] (3.24)

Now from (3.16) and (3.24) we have
\[ u(x,t) = K \exp \left( \frac{-2(e^{2\gamma(T-t)} + 1)x}{\gamma(e^{2\gamma(T-t)} - 1) + k(e^{2\gamma(T-t)} - 1)^2} - \frac{\theta k(\gamma + k)t}{\sigma^2} \right) \times \frac{(\gamma + k)(\gamma + 1) + 2\gamma}{\gamma(e^{2\gamma(T-t)} - 1)} \] (3.25)

Using now the final condition for the Zero-Coupon bond we find
\[ K = 2\gamma \frac{2\theta k}{\sigma^2} \exp \left( \frac{\theta k(\gamma + k)T}{\sigma^2} \right) \]

such that (3.25) becomes
\[ u(x,t) = \exp \left( \frac{-2x(e^{2\gamma(T-t)} + 1)}{(\gamma + k)(e^{2\gamma(T-t)} - 1) + 2\gamma} \right) \exp \left( \frac{(\gamma + k)(T-t) 2\theta k}{2 \sigma^2} \right) \times \frac{2\gamma}{(\gamma + k)(e^{2\gamma(T-t)} - 1) + 2\gamma} \] (3.26)

which is the analytic solution for the Zero-Coupon bond price under the CIR model.

### 3.2 The constant elasticity of variance (CEV) model

Empirical analysis of observed option prices, and observed phenomena like the volatility smile, led researchers to formulate option pricing models that included non-constant volatility. In response to observations of an inverse relationship between share price and share price volatility, documented by Fischer Black (1975), the Constant Elasticity of Variance (CEV) model was derived. Cox (1996) suggests that the CEV model is the simplest way to describe such an inverse relationship. In fact,
3.2 The constant elasticity of variance (CEV) model

The CEV model was derived after a direct request from Fischer Black to John Cox, for a share price evolution model that includes an inverse dependence of volatility and the share price, as described in Cox (1996).

The CEV process is a generalization of the CIR model introduced in 3.1, it will seen how in a while, but whereas the CIR is widely used in modelling interest rate the CEV model appear dealing with equity markets and in option pricing.

Let $S_t = \{S_t : t \geq 0\}$ be the stock price governed by the CEV stochastic differential equation

$$
\begin{align*}
\frac{dS_t}{S_t} = \mu S_t dt + \sigma S_t^{\beta/2} dW_t
\end{align*}
$$

with $W_t$ a standard Wiener process, $\beta < 2$, $\beta - 2$ is the elasticity and $\sigma > 0$ a constant.

The return variance $\kappa(S_t, t)$ with respect to prices $S_t$ has the following relationship

$$
\frac{d\kappa(S_t, t)}{\kappa(S_t, t)/S_t} = \beta - 2
$$

it follows then integrating and exponentiating both side that

$$
\kappa(S_t, t) = \sigma^2 S_t^{\beta-2}
$$

If $\beta = 2$ then the elasticity is 0 and the prices are lognormally distributed and we recover the standard Black-Scholes model.

If $\beta = 1$ we have the well known Cox-Ingersoll-Ross (CIR) model.

We have to specify that studies on the CEV with $\beta > 2$ has been done by Emanuel & MacBeth.

3.2.1 CEV transition density function

Since exploiting the CEV transition density function is not an easy task, via Ito’s lemma, we will transform the CEV into the CIR process. We will apply then the results we have already obtained and then transform everything back to the CEV process in order to retrieve the density function of the CEV process.

Let us see how in details.

Let us now define $f(x) = x^{2-\beta}$. Therefore by I-D’s lemma we have

$$
\begin{align*}
\frac{dX_t}{dS_t} &= \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \sigma^2 S_t^{\beta} \frac{\partial^2 f}{\partial S^2} dt = \\
&= \left[ (2 - \beta) \mu S_t S_t^{1-\beta} + \frac{\sigma^2 S_t^{3}}{2} (2 - \beta)(1 - \beta) S_t^{1-\beta} \right] dt + \sigma S_t^{\beta/2} (2 - \beta) S_t^{1-\beta} dW_t = \\
&= \left[ (2 - \beta) \mu X_t + \frac{\sigma^2}{2} (2 - \beta)(1 - \beta) \right] dt + \sigma (2 - \beta) S_t^{1-\beta} dW_t = \\
&= \left[ (2 - \beta) \mu X_t + \frac{\sigma^2}{2} (2 - \beta)(1 - \beta) \right] dt + \sigma (2 - \beta) \sqrt{X_t} dW_t
\end{align*}
$$
For the sake of simplicity we will now change the notation

\[
\begin{align*}
    k &= (\beta - 2) \mu \\
    \theta &= \frac{\sigma^2}{2\mu} (\beta - 1) \\
    \tilde{\sigma} &= (2 - \beta) \sigma
\end{align*}
\]

in order to recover the standard form of the CIR process studied in section 3.1.

We have therefore retrieved the process \( X = \{X_t : t \geq 0 \} \) solution to the CIR SDE

\[
\begin{align*}
    dX_t &= k(\theta - X_t)dt + \tilde{\sigma} \sqrt{X_t}dW_t \\
    X_0 &= x^0
\end{align*}
\]

Applying now theorem 2.2.10 and using the inverse Laplace transform, see again 3.1 for details, we can recover the transition density for the CIR process.

\[
p_{CIR}(x, y, \tau) = \frac{2k e^{k(2\mu + 1)\tau}}{\sqrt{x}} \exp\left( \frac{-2k(x + e^{k\tau}y)}{\sigma^2(\beta - 1)} \right) I_\nu\left( \frac{2k\sqrt{xy}}{\sigma^2 \sinh(\frac{k\tau}{2})} \right),
\]

with \( \nu = \frac{2\theta}{\sigma^2} - 1 \) and

\[
I_\nu(x) = \frac{x}{2} \sum_{n \geq 0} \frac{(x/2)^{2n}}{n!\Gamma(\nu + n + 1)}
\]

a modified Bessel function of the first kind of order \( \nu \).

Since our goal was to find and explicit formula for the CEV transition density, we have now to use again the transformation in order to recover the density of the original process.

Let us just recall first of all that \( X = f(S) = S^{2-\beta} \) and therefore \( S = f^{-1}(X) = g(X) = X^{\frac{1}{2-\beta}} \).

Since the function is clearly increasing, we know from probability 1.0.1 that this is enough to allow us to invert the transformation, we can easily evaluate the transition density to be

\[
p_{CEV}(x, y, t) = \left| \frac{\partial f(y)}{\partial y} \right| p_{CIR}(f(x), f(y), t)
\]

Recalling that we originally renamed all the parameters, see (3.27), we can evaluate the probability transition density for the CEV process as follows

\[
P_{CEV}(x, y, \tau) = \frac{8\mu e^{(2-3)\mu} \tau}{\sigma^2(\beta - 2)} \left( \frac{x}{y} \right) \left( \frac{\beta - 1}{\beta - 2} \right)
\]

\[
\exp\left\{ -2\mu(x^{2-\beta} + e^{\beta \mu}y^{2-\beta}) \right\} I_{\frac{1}{\beta - 2}}\left( \frac{2\mu(xy)^{2-\beta}}{\sigma^2(\beta - 2) \sinh(\frac{(\beta - 2)\mu t}{2})} \right)
\]

Lemma 3.2.1 (Feller,1951). Let \( p(x, y, t) \) the transition density function for \( x \) and \( t \) conditional to \( y \)
of a given process. Then an explicit fundamental solution of the parabolic problem

\[(P(x,t))_t = (a x P(x,t))_{xx} - ((b x + h) P(x,t))_x, \quad 0 < x < \infty\]

is given by

\[p(x,y,t) = \frac{b}{a(e^{bt} - 1)} \left( \frac{e^{-bx}}{y} \right)^{\frac{a}{2b}} \exp \left[ \frac{-b(x + ye^{bt})}{a(e^{bt} - 1)} \right] I_{\frac{1}{2}} \left( \frac{2b}{a(1-e^{bt})} (e^{bt}xy)^{\frac{1}{2}} \right)\]

where \(I_k(x)\) is the modified Bessel function of the first kind of order \(k\).

We can check that the transition probability density is the same as the one derived thorough Feller’s lemma 3.2.1 with appropriate parameters. See Feller [Fel51] for details.

### 3.2.2 CEV option pricing model

Let now be \(S = \{S_t : t \geq 0\}\) the stock price driven by the CEV stochastic differential equation

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sigma S_t^{\beta/2} dW_t \\
    S_0 &= s
\end{align*}
\]

Let furthermore \(\Pi\) be a portfolio and \(U(S_t)\) denote the value of an option in the portfolio \(\Pi\) and be \(\Delta\) the quantity of the stock in \(\Pi\). That is

\[\Pi = U - \Delta S\]

An infinitesimal change in the portfolio in a time interval \(dt\), setting \(\Delta = \frac{\partial U}{\partial S}\) (we chose \(\Delta\) in such a way in order to get rid of the stochastic terms!), is given by

\[d\Pi = \left\{ \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial S} dS + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} dS^2 - \frac{\partial U}{\partial S} dS \right\} = \left( \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} \sigma^2 S^\beta \right) dt\]

Since it does not appear any stochastic term the value of the portfolio is certain. Thus in order to preclude arbitrage the payoff must be equal to \(\Pi r dt\), then

\[\left( \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} \sigma^2 S^\beta \right) dt = \left( U - \frac{\partial U}{\partial S} S \right) r dt\]

We thus have the following PDE

\[\frac{\partial U}{\partial t} + st \frac{\partial U}{\partial S} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} \sigma^2 S^\beta - r U = 0 \quad (3.29)\]

Let us now assume for instance that \(U(S_t, t) =: C(S_t, t)\) is a call option. Therefore we give as boundary condition \(C_T = \max(S_T - K, 0)\) with \(K\) the strike price. We will not go deeper into financial details. For any doubt on financial topics we refer to Shreve [Shr04] and Cvitanić and Zapatero [CZ04].
We thus have to solve the following PDE

\[
\begin{aligned}
\frac{\partial C}{\partial t} + sr \frac{\partial C}{\partial S} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^\beta - rC &= 0 \\
C_T &= \max(S_T - K, 0)
\end{aligned}
\] (3.30)

The standard approach in literature is to rewrite everything in terms of the logarithm of the spot price \(x = \ln(S)\) and evaluating thus

\[
\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x} \frac{1}{S}, \quad \frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2} \frac{\partial^2 C}{\partial x^2}
\]

Equation (3.32) can thus be written as

\[
\begin{aligned}
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial C}{\partial x} &+ \frac{1}{2} \sigma^2 \sigma^2 e^x (\beta - 2) - rC = 0 \\
C_T &= \max(e^{x_T} - K, 0)
\end{aligned}
\] (3.31)

Then, in analogy with the Black–Scholes approach a solution of the form

\[
C_t(x, t) = e^x P_1(x, t) - K e^{-r(T-t)} P_2(x, t)
\]

is guessed. See Chuang, Hsu, and Lee [CHL10], Chen and Lee [CL10], and Hsu, Lin, and Lee [HLL08] for details.

We will now follow a different approach. As already done in section 3.2.1 we will transform the CEV process into the CIR process. Since it has done previously we will now avoid all the computation and we will start from the process \(S = \{X_t : t \geq 0\}\) satisfying the stochastic differential equation

\[
\begin{aligned}
dX_t &= k(\theta - X_t)d\tau + \tilde{\sigma}\sqrt{X_t}dW_{\tau} \\
X_0 &= x_0
\end{aligned}
\]

We will consider again a portfolio \(\Pi\) composed as before, just this time the stock price is driven by the CIR process.

We thus have the following

\[
d\Pi = \left( \frac{\partial U}{\partial t} + \tilde{\sigma}^2 \frac{\partial^2 U}{\partial X^2} X \right) dt = \left( U - \frac{\partial U}{\partial X} X r \right) dt
\]

Changing the variable \(t := T - t\) in order to have the forward Cauchy problem with initial value \(\max(X_0 - K)\), in analogy with (3.32) we get the following

\[
\begin{aligned}
\frac{\partial C}{\partial t} &= X_T \frac{\partial C}{\partial X} + \frac{\sigma^2}{2} \frac{\partial^2 C}{\partial X^2} X - rC \\
C_0 &= \max(X_0 - K, 0)
\end{aligned}
\] (3.32)
3.2 The constant elasticity of variance (CEV) model

Ignoring for the moment the Cauchy problem we merely focus on the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - ru$$  (3.33)

We would like to compute the Lie algebra of equation (3.33). Proceeding as usual we can find seven determining equations given the general infinitesimal generator

$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u}$$

$$\begin{align*}
\xi_u & = 0 \quad (3.34a) \\
\tau_u & = 0 \quad (3.34b) \\
\phi_{uu} & = 0 \quad (3.34c) \\
\tau_x & = 0 \quad (3.34d) \\
2x\sigma^2\phi_{x,u} + 2\xi - 2rx\xi_x - x\sigma^2\xi_{x,x} + 2rx\tau_x + 2\xi & = 0 \quad (3.34e) \\
2\phi_t - 2ru\phi_x - 2rx\phi_x - x\sigma^2\phi_{x,x} + 2ru\tau_x + 2r\phi & = 0 \quad (3.34f) \\
-2x\xi_x + x\tau_t + \xi & = 0 \quad (3.34g) \\
\end{align*}$$

The previous system leads to the most general form of the coefficients of the infinitesimal symmetry of equation (3.33)

$$\begin{align*}
\xi(x, t) & = e^{-rt}(e^{2rt}k_1 + k_2)x \\
\tau(x, t) & = \frac{e^{rt}k_1 - e^{-rt}k_2 + kr}{r} \\
\phi(x, t) & = e^{-rt}k_2 + k_4 = \frac{e^{rt}k_1(2rx + \sigma^2)}{\sigma^2} + \alpha(x, t) \\
\end{align*}$$  (3.35)

where $\xi$, $\tau$ and $\phi$ are the coefficients of the general infinitesimal generator

$$v = \xi(x, t)\partial_x + \tau(x, t)\partial_t + u\phi(x, t)\partial_u$$

and $\alpha(x, t)$ a general solution of equation (3.33).

The system (3.35) allows us to determine a basis for the Lie algebra to be

$$\begin{align*}
v_1 & = e^{rt}x\partial_x + \frac{e^{rt}}{r} \partial_t - \frac{(2rx + \sigma^2)}{\sigma^2} e^{rt} u\partial_u \\
v_2 & = e^{-rt}x\partial_x - \frac{-e^{-rt}}{r} \partial_t + e^{-rt} u\partial_u \\
v_3 & = \partial_t \\
v_4 & = u\partial_u \\
\end{align*}$$  (3.36)

The commutator table is thus
We are looking for those generators \( v \) who preserves the boundary conditions. Therefore we require

\[
v[x]_{x=0} = 0, \quad v[t]_{t=0} = 0, \quad v[x-K]_{x=K} = 0
\]

Those invariance lead us to the following system

\[
\begin{cases}
\tau(0) = 0 \Rightarrow k_1 - k_2 + k_3r = 0 \\
\xi(K) = 0 \Rightarrow e^{rt}Kk_1 + e^{-rt}Kk_2 = 0
\end{cases}
\] (3.37)

Since the last condition holds for any \( t \) the following easily follows Those invariance lead us to the following system

\[
\begin{cases}
\tau(0) = 0 \Rightarrow k_1 - k_2 + k_3r = 0 \\
k_1 = -k_2 \\
k_3 = \frac{2k_2}{r}
\end{cases}
\] (3.38)

We can now reformulate the system (3.35) as

\[
\begin{align*}
\xi(x, t) &= k_2(e^{-rt} - e^{rt})x \\
\tau(x, t) &= \frac{k_2}{r}(2 - e^{-rt} - e^{rt}) \\
\phi(x, t) &= k_4 + k_2(e^{-rt} + e^{rt}(\frac{2rx + \sigma^2}{\sigma^2}))
\end{align*}
\] (3.39)

We would like to recover a fundamental solution. Thus we prescribe \( u(x, 0) = \delta(x - y) \) as initial datum, with \( \delta \) the Dirac mass. In order to such a condition to hold we require

\[
\phi(x, 0)\delta(x - y) = \xi(x, 0)\delta'(x - y)
\]

Thus we get the following

\[
\begin{cases}
\xi(y, 0) = 0 \\
\phi(y, 0) = -\frac{\partial\xi}{\partial x}(y, 0)
\end{cases}
\]

Substituting now the last equation into (3.39) we get

\[
\phi(y, 0) = 0 \Rightarrow k_4 = -k_2\left(\frac{2\sigma + 2ry}{\sigma^2}\right)
\]
3.2 The constant elasticity of variance (CEV) model

Therefore we get the following coefficients

\[
\begin{align*}
\xi(x, t) &= k_2(e^{-rt} - e^{rt}) x \\
\tau(x, t) &= \frac{k_2}{r}(2 - e^{-rt} - e^{rt}) \\
\phi(x, t) &= k_2 \left( e^{-rt} + e^{rt} \left( \frac{2rx + \sigma^2}{\sigma^2} \right) - \frac{2\sigma^2 + 2ry}{\sigma^2} \right)
\end{align*}
\] (3.40)

We then compute the invariants for the action of

\[
v = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + u\phi \frac{\partial}{\partial u}
\] (3.41)

with \(\xi, \tau\) and \(\phi\) given as in (3.40).

Equivalently

\[
v = k_2\left( -v_1 + v_2 + \frac{2}{r}v_3 - 2\frac{\sigma^2 + yr}{\sigma^2}v_4 \right)
\] (3.42)

or in explicit form

\[
v = k_2\left[ (e^{-rt} - e^{rt}) x \partial_x + \left( \frac{2 - e^{-rt} - e^{rt}}{r} \right) \partial_t + \left( e^{-rt} + e^{rt} \left( \frac{2rx + \sigma^2}{\sigma^2} \right) - \left( \frac{2\sigma^2 + 2ry}{\sigma^2} \right) \right) u \partial_u \right]
\] (3.43)

Invariance of the action are given by solving the equation \(v[\eta] = 0\).

We can find invariance by solving the characteristic system

\[
\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}
\] (3.44)

Integrating the first and second part of equation (3.44) we get the invariant

\[
\eta = \eta(x, t) = \frac{e^{-rt}(1 - e^{rt})^2}{x} = \frac{2(\cosh(rt) - 1)}{x}
\] (3.45)

In order to find the other invariant we replace

\[
x = \frac{2(\cosh(rt) - 1)}{\eta}
\]

noting that \(\eta\) is constant for all solutions of the characteristic system. Thus we get the second invariant

\[
v = \exp\{h(x, t)\} u(x, t) = \exp\left\{ \frac{r \left[ -2y + t\sigma^2 + 2e^{rt}(y - t\sigma^2) + e^{2rt}(2x + t\sigma^2) \right]}{\sigma^2(e^{rt} - 1)^2} \right\} u(x, t)
\]

Thus since \(\eta\) is a basis for the invariants it follows that \(v = \Psi(\eta)\) for a generic function \(\Psi\). Thus we have the following general form of a solution \(u(x, t)\)

\[
u(x, t) = \exp\{-h(x, t)\} \Psi(\eta)
\] (3.46)
Substituting now equation (3.46) into equation (3.33) we get that $\Psi(\eta)$ has to satisfy

$$ -r^2 x^3 (x + 2y - 2y \cosh(rt))\Psi(\eta) + 16x\sigma^2 \left[ rx - \sigma^2 + \sigma^2 \cosh(rt) \right] \sinh^4(\frac{rt}{2})\Psi'(\eta) + $$

$$ + 64\sigma^4 \sinh^8(\frac{rt}{2})\Psi''(\eta) = 0 $$

(3.47)

### 3.2.3 Mapping

Recalling theorem 1.4.3 we can see that since the Lie algebra of equation (3.33) is spanned by four vector fields we can find a one-to-one mapping to the partial differential equation

$$ v_y = v_{\tau \tau} - \frac{A}{y^2}v $$

(3.48)

whose Lie algebra is spanned by

$$ \begin{align*}
  w_1 &= \tau y \partial_y + \tau^2 \partial_{\tau} - \left( \frac{1}{2} y^2 + \frac{1}{2} \tau \right) v \partial_v \\
  w_2 &= \partial_{\tau} \\
  w_3 &= y \partial_y + 2\tau \partial_{\tau} \\
  w_4 &= v \partial_v
\end{align*} $$

(3.49)

with commutator table

<table>
<thead>
<tr>
<th></th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0</td>
<td>$-w_3 + \frac{1}{2}w_4$</td>
<td>$-2w_1$</td>
<td>0</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_3 - \frac{1}{2}w_4$</td>
<td>0</td>
<td>$2w_2$</td>
<td>0</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$2w_1$</td>
<td>$-2w_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We recall that the backward parabolic equation

$$ u_t = xu_x + \sigma^2 \frac{x^2}{2} u_{xx} - ru $$

(3.50)

has a four dimensional Lie algebra spanned by

$$ \begin{align*}
  v_1 &= e^{rt}x \partial_x + e^{rt} \tau \partial_{\tau} - \left( \frac{2rx + \sigma^2}{\sigma^2} \right) e^{rt} u \partial_u \\
  v_2 &= e^{-rt}x \partial_x - \frac{e^{-rt} \tau}{\tau} \partial_{\tau} + e^{-rt} u \partial_u \\
  v_3 &= \partial_{\tau} \\
  v_4 &= u \partial_u
\end{align*} $$

(3.51)

and commutator table
3.2 The constant elasticity of variance (CEV) model

<table>
<thead>
<tr>
<th></th>
<th>v_1</th>
<th>v_2</th>
<th>v_3</th>
<th>v_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>v_1</td>
<td>0</td>
<td>-(\frac{2\alpha}{r}) + 2v_4</td>
<td>rv_1</td>
<td>0</td>
</tr>
<tr>
<td>v_2</td>
<td>2(v_1-r)</td>
<td>0</td>
<td>-rv_2</td>
<td>0</td>
</tr>
<tr>
<td>v_3</td>
<td>-rv_1</td>
<td>rv_2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>v_4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If we consider the Lie subalgebra \(\mathfrak{h}^{(23)}\) spanned by \(v_2\) and \(v_3\) (equivalently by \(w_1\) and \(w_3\)) and considering the scaling \(w_i = \alpha_i \tilde{w}_i\) with \(i = 2, 3\), the the commutators arising from \(\{\tilde{w}_i\}_i\) are the same as the commutators in table (3.2.3) arising from \(\{v_i\}_i\) if

\[
\alpha_2 = -\frac{8r}{\sigma^2}, \quad \alpha_3 = \frac{2}{r}
\]

(3.52)

This suggests seeking a one-to-one point transformation mapping of the PDE (3.43) to the target PDE (3.48) such that it maps each \(v_i\) to the corresponding \(\tilde{w}_i\) with

\[
w_2 = -\frac{8r}{\sigma^2} \tilde{w}_2, \quad w_3 = \frac{2}{r} \tilde{w}_3
\]

(3.53)

In particular we look for a transformation of the form

\[
\begin{align*}
g &= \alpha(x, t) \\
\tau &= \beta(t) \\
v &= \gamma(x, t, u)
\end{align*}
\]

(3.54)

Such a transformation has to satisfy conditions

\[
\begin{align*}
v_j \alpha(x, t) &= \tilde{w}_j \beta(y, \tau, v) = (\alpha, \beta, \gamma), & v_j \beta(t) &= \tilde{w}_j \gamma(y, \tau, v) = (\alpha, \beta, \gamma) \\
v_j \gamma(x, t, u) &= \tilde{w}_j \gamma(y, \tau, v) = (\alpha, \beta, \gamma)
\end{align*}
\]

(3.55)

with \(j = 2, 3\) System (3.55) read extensively

\[
\begin{align*}
e^{-rt}x\alpha_x + \frac{e^{rt}r}{\alpha_t} \alpha_t &= 0 \quad (3.56a) \\
\alpha_t &= \frac{r}{2} \alpha \quad (3.56b) \\
-e^{-rt} \frac{r}{\sigma^2} \beta_t &= \frac{8r}{\sigma^2} \beta_t \quad (3.56c) \\
\beta_t &= r \beta \quad (3.56d) \\
e^{-rt}xu\gamma_x - e^{-rt} \frac{r}{\alpha} u\gamma_t + e^{-rt} u\gamma_t &= 0 \quad (3.56e) \\
u\gamma_t &= 0 \quad (3.56f) \\
\end{align*}
\]

(3.56g)
Solutions of the previous system are

\[
\begin{align*}
    y &= \alpha(x,t) = \exp\left\{\frac{\sigma^2}{2}t\right\}\sqrt{x} \\
    \tau &= \beta(t) = -\frac{\sigma^2}{8}\exp\{rt\} \\
    v &= \frac{y}{\bar{y}}
\end{align*}
\]

(3.57)

We would like to stress that all the constants were chosen appropriately after careful computation.

Since estimation are tough to perform we prefer to proceed step by step. We first of all apply the following change of variables

\[
\begin{align*}
    x &= \frac{y^2\sigma^2}{8} \\
    t &= \log\left(\frac{8\sigma^2}{\sigma^2 r}\right) \\
    u(x,t) &= \tilde{v}(y(x,t), \tau(x,t)) = \tilde{v}\left(\exp\left\{\frac{\sigma^2}{2}t\right\}\sqrt{x}, \exp\{rt\}\frac{\sigma^2}{8}\right)
\end{align*}
\]

(3.58)

By the standard chain rule we can express the partial derivatives of \(u\) w.r.t. variables \((x,t)\) in terms of partial derivative of \(\tilde{v}\) w.r.t. variables \((y,\tau)\) as

\[
\begin{align*}
    u_t &= \sqrt{x}\tilde{v}_y + \frac{\sigma^2}{8}\exp\{rt\}\tilde{v}_\tau \\
    u_x &= \frac{1}{2}\sqrt{x}\tilde{v}_y \\
    u_{xx} &= \frac{1}{4}\tilde{v}_y + \frac{1}{4\sigma^2}\exp\{rt\}\tilde{v}_y
\end{align*}
\]

(3.59)

Substituting now equation (3.59) into equation (3.50) we get the following

\[
\tilde{v}_\tau = \tilde{v}_{yy} - \frac{4}{y}\tilde{v}_y - \frac{1}{\tau}\tilde{v}
\]

(3.60)

Applying now the change of variable

\[
\tilde{v}(y,\tau) = \frac{y^2\sigma^2}{8\tau}v(y,\tau)
\]

(3.61)

and computing the partial derivatives we get

\[
\begin{align*}
    \tilde{v}_\tau &= -\frac{1}{\tau^2}\frac{y^2\sigma^2}{8}v + \frac{\sigma^2^2}{8\tau^2}v_\tau \\
    \tilde{v}_y &= \frac{2\sigma^2}{8\tau}v + \frac{\sigma^2^2}{8\tau^2}v_y \\
    \tilde{v}_{yy} &= \frac{2\sigma^2}{8\tau^2}v + \frac{4\sigma^2^2}{8\tau^2}v_y + \frac{\sigma^2^4}{8\tau^2}v_{yy}
\end{align*}
\]

(3.62)

We thus get the equation

\[
v_\tau = v_{yy} - \frac{6}{y^2}v
\]

(3.63)

This equation has a fundamental solution of the form

\[
p(y,\hat{y},\tau) = \frac{\sqrt{y\hat{y}}}{2\tau}\exp\{-\frac{y^2 + \hat{y}^2}{4\tau}\}I_\frac{y}{2\tau}\left(\frac{y\hat{y}}{2\tau}\right)
\]

(3.64)
3.3 The Stochastic-alpha-beta-rho (SABR) model

In this section we will study a model where the volatility of stocks return is assumed to be stochastic. We will give particular emphasis to the model proposed by Heston. For some further notions on stochastic volatility model we refer to Brigo and Mercurio [BM06] and Joshi [Jos08].

The SABR model is a stochastic volatility model which attempts to capture the volatility smile in derivatives markets. The CEV, studied in section ?? is an important ingredient of such a model.

Let us assume that the stock price \( S = \{ S_t : t \geq 0 \} \) is driven by the process under the risk neutral measure

\[
dS_t = rS_t dt + \nu(t) \frac{\beta}{2} S_t dW^1_t
\]

where \( \mu(t) \) is the instantaneous drift of stock return, \( \nu_t \) the variance and \( W^1_t \) a standard Wiener process. We assume further that the variance \( V = \{ V_t : t \geq 0 \} \) follows

\[
\nu_t = \mu_v(S, \nu, t) dt + \sigma_v \nu^\alpha \beta(S, \nu, t) dW^2_t
\]

where \( W^2_t \) is again a standard Wiener process with \( dW^1_t dW^2_t = \rho dt, -1 \leq \rho \leq 1 \), \( \eta \) is the volatility of the volatility and \( \rho \) is the correlation between the two Wiener processes. It is often considered \( \beta = 1 \) in equation (3.65) and \( \alpha = \frac{1}{2} \), anyway we can always pass from a process to the other via Ito’s lemma.

**Lemma 3.3.1** (Ito’s lemma). Let \( f \) be a function with independent variables \( (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ \). Let us suppose that \( X_i \) follows the Ito process

\[
dX^i_t = a_i(x, t) dt + b_i(x, t) dW^i_t
\]

where as usual the \( W^i_t \) are standard Wiener processes with correlation coefficient \( dW^i_t dW^j_t = \rho_{ij} dt \).

Then

\[
df = \left( \sum_{i} \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} \right) dt + \sum_{i} \frac{\partial f}{\partial x_i} b_i dW^i_t
\]

Since we are dealing with two sources of randomness we have to hedge both the stock price and the volatility. Thus we can set up a portfolio \( \Pi \) with the option we are to price, denoted by \( U(S, \nu, t) \), a quantity \(-\Delta\) of the stock plus a quantity \(-\Delta_1\) of another asset with value \( U_1(S, \nu, t) \). Therefore we have

\[
\Pi = U(S, \nu, t) - \Delta S - \Delta_1 U_1(S, \nu, t)
\]

Applying now lemma 3.3.1 with \( N = 2 \) we get

\[
dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial \nu} d\nu + \left( \frac{\partial U}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 U}{\partial S^2} + \rho \nu S \frac{\partial^2 U}{\partial \nu \partial S} + \frac{1}{2} \nu \beta^2 \frac{\partial^2 U}{\partial \nu^2} \right) dt
\]

\[
dU_1 = \frac{\partial U_1}{\partial S} dS + \frac{\partial U_1}{\partial \nu} d\nu + \left( \frac{\partial U_1}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 U_1}{\partial S^2} + \rho \nu S \frac{\partial^2 U_1}{\partial \nu \partial S} + \frac{1}{2} \nu \beta^2 \frac{\partial^2 U_1}{\partial \nu^2} \right) dt
\]
\[ d\Pi = dU - \Delta dS - \Delta_1 dU_1 = \]
\[ = \left( \frac{\partial U}{\partial t} + \nu \frac{\partial^2 U}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U}{\partial \nu^2} \right) dt \]
\[ - \Delta_1 \left( \frac{\partial U_1}{\partial t} + \nu \frac{\partial^2 U_1}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U_1}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U_1}{\partial \nu^2} \right) dt \]
\( (3.68) \)

Setting now
\[ \frac{\partial U}{\partial S} - \Delta_1 \frac{\partial U_1}{\partial S} = 0 \]

and
\[ \frac{\partial U}{\partial \nu} - \Delta_1 \frac{\partial U_1}{\partial \nu} = 0 \]

we can eliminate all randomness from the portfolio. Thus from arbitrage assumption we get

\[ d\Pi = dU - \Delta dS - \Delta_1 dU_1 = \]
\[ = \left( \frac{\partial U}{\partial t} + \nu \frac{\partial^2 U}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U}{\partial \nu^2} \right) dt \]
\[ - \Delta_1 \left( \frac{\partial U_1}{\partial t} + \nu \frac{\partial^2 U_1}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U_1}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U_1}{\partial \nu^2} \right) dt \]
\[ = r\Pi dt = r(U - \Delta S - \Delta_1 U_1) dt \]

We can collect now all \( U \) terms on the left side and all the \( U_1 \) terms on the right side ending up with

\[ \frac{\partial U}{\partial t} + \nu S^2 \frac{\partial^2 U}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U}{\partial \nu^2} + rS \frac{\partial U}{\partial S} - rU = \]
\[ = \frac{\partial U_1}{\partial t} + \nu S^2 \frac{\partial^2 U_1}{\partial S^2} + \rho \nu \beta S \frac{\partial^2 U_1}{\partial S \partial \nu} + \frac{1}{2} \eta^2 \nu \beta^2 \frac{\partial^2 U_1}{\partial \nu^2} + rS \frac{\partial U_1}{\partial S} - rU_1 \]
\( (3.70) \)

Since the left hand side of equation (3.70) is a function of \( U \) alone and the right hand side depends on \( U_1 \). The only way this can be true is that both terms are equal to a function \( f(S, \nu, t) \).

### 3.3.1 The Heston model

The Heston model has been proposed by Heston in 1993 in Heston [Hes93]. It is a particular case of the general model introduced in the previous section. In particular it correspond to the case of

\[ \mu_\nu(s, \nu, t) = \kappa(\theta - \nu_t), \quad \beta = 1, \quad \alpha = \frac{1}{2} \]
in equation (3.66). Therefore we are dealing with

\[
\begin{aligned}
dS_t &= rS_t dt + \sqrt{\nu_t} S_t dW^1_t \\
d\nu_t &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW^2_t
\end{aligned}
\]

with \(dW^1_t dW^2_t = \rho dt\), \(\kappa\) the speed of reversion of \(\nu_t\) and \(\theta\) the long-term mean of volatility process.

Recalling now what we said at the end of the previous section we get

\[
\begin{aligned}
\frac{\partial U}{\partial t} + \nu S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma \nu S \frac{\partial^2 U}{\partial S \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} - rU + (\kappa (\theta - \nu_t) - \lambda(S, \nu, t)) \frac{\partial U}{\partial \nu} &= 0 \\
\end{aligned}
\]

If we consider the case of a European call option we have the following boundary condition

\[
\begin{aligned}
U(S, \nu, T) &= \max(S_T - K, 0) \\
U(0, \nu, t) &= 0 \\
\frac{\partial U}{\partial S}(\infty, \nu, t) &= 1 \\
rS \frac{\partial U}{\partial S}(S, 0, t) + \kappa \theta \frac{\partial U}{\partial \nu}(S, 0, t) - rU(S, 0, t) + \frac{\partial U}{\partial t}(S, 0, t) &= 0 \\
U(S, \infty, t) &= S
\end{aligned}
\]

Usually in analogy to the Black-Scholes model a solution of the form

\[
U(S, \nu, t) = SP_1 - Ke^{-r(T-t)}P_2
\]

is guessed. As done in section ?? we will have a non standard approach using Lie’s algebras.

We will consider first of all the much simpler case of uncorrelated Wiener processes. Then we have to perform some change of variables in order to consider a easier parabolic problem.

Let us in fact perform a change of variable in order to have a well-posed forward parabolic problem and setting furthermore

\[
t := T - t, \quad X = \ln(S), \quad U(t, S, \nu) = u(t, \ln(S), \nu)
\]

by the chain rule we have

\[
\begin{aligned}
U_S &= u_x x_S = u_x \frac{1}{S} \\
U_{SS} &= \frac{1}{S} u_{xx} - \frac{1}{S^2} u_x \\
U_t &= -u_t
\end{aligned}
\]

substituting now into equation (3.72) we get the following problem

\[
u_t = \frac{\nu}{2} u_{xx} + \left( r - \frac{\nu}{2} \right) u_x + \frac{\sigma^2}{2} \nu u_{\nu \nu} + \kappa (\theta - \nu_t) u_\nu - ru
\]
We apply now the Fourier transform to the function \( u(t, x, \nu, t) \) in the variable \( x \)

\[
\hat{u}(t, \xi, \nu) := \mathcal{F}[u(t, x, \nu)](\xi) = \int_{\mathbb{R}} u(t, x, \nu)e^{-i\xi x}dx
\]

Recalling now the usual differentiation rule for the Fourier transform

\[
\hat{D}^\alpha u = i|\alpha|\xi^\alpha \hat{u}
\]

we have that the only derivatives that change are

\[
\begin{align*}
\hat{D}_x^2 u &= -\xi^2 \hat{u} \\
\hat{D}_x u &= i\xi \hat{u}
\end{align*}
\]

Thus we get the following parabolic problem

\[
\hat{u}_t = \frac{\sigma^2}{2} \nu \hat{u}_{\nu\nu} + \kappa(\theta - \nu)\hat{u}_\nu - \nu \mu \hat{u} + \alpha \hat{u}
\]  

(3.77)

Setting now \( V = e^{-\alpha t} \hat{u} \) we get eventually the following PDE

\[
V_t = \frac{1}{2} \sigma^2 \nu V_{\nu\nu} + \kappa(\theta - \nu) V_\nu - \mu V
\]

(3.78)

with

\[
\mu = \frac{1}{2}(\xi^2 - i\xi), \quad \alpha = r(i\xi - 1)
\]

We would like to recover a fundamental solution \( p(x, y, t) \) of equation (3.78). In fact given such a fundamental solution we have that

\[
\hat{U}(\xi, \nu, t) = e^{\alpha(T-t)} \int \hat{\phi}(\xi, y)p(T-t, \nu, y)dy
\]

(3.79)

with \( \hat{\phi} \) the Fourier transform of an appropriate initial datum \( \phi \).

We can now see that equation (3.78) is of the form (2.28). Applying thus theorem 2.2.11 we can obtain a fundamental solution of the form (2.29)

\[
p(\nu, y, t) = \frac{\sqrt{A\nu y}e^{-(F(\nu) - F(y))/(\sigma^2)}}{\sigma^2 \sinh(\sqrt{A}t/2)} \exp \left( -\frac{Bt}{\sigma^2} - \frac{\sqrt{A}(\nu + y)}{\sigma^2 \tanh(\sqrt{A}t/2)} \right) \\
\left( C_1(y)I_n \left( \frac{2\sqrt{A\nu y}}{\sigma^2 \sinh(\sqrt{A}t/2)} \right) + C_2(y)I_{-n} \left( \frac{\sqrt{A\nu y}}{\sigma \sinh(\sqrt{A}t/2)} \right) \right)
\]

(3.80)

with \( n = \frac{2\sqrt{A^2 + \sigma^2}}{\sqrt{2}\sigma^2} \), the constants \( A, B \) and \( C \) such that \( f = \kappa(\theta - \nu) \) solution of the second Riccati equation

\[
\sigma \nu f' - \sigma f + \frac{1}{2} f^2 + 2\sigma \mu \nu^2 = \frac{1}{2} Av^2 + Bv + C, \quad \text{with} \ A > 0
\]

(3.81)

and \( F'(x) = \frac{f(x)}{x} \).
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Thus given an appropriate initial datum \( U(0, S, \nu) = \phi(S, \nu) \) the evaluation of equation

\[
\begin{aligned}
\frac{\partial U}{\partial t} + \frac{\nu^2 S^2}{2} \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + r \frac{\partial U}{\partial S} - rU + f(\nu) \frac{\partial U}{\partial \nu} &= 0 \\
\end{aligned}
\]

is reduced to the evaluation of the integral (3.79)

\[
\hat{U}(\xi, \nu, t) = e^{\kappa(T-t)} \int \hat{\phi}(\xi, y)p(T-t, \nu, y)dy
\]

with \( p(T-t, \nu, y) \) of the form (3.80).

We thus get

\[
\kappa + 2\sigma^2 \mu = A, \quad B = -\kappa^2 \theta, \quad C = \frac{\sigma^2}{2} \kappa \theta + \frac{\kappa^2}{2} \theta^2, \quad F(x) = \kappa \theta \ln(x) - \kappa x
\]
Appendix A

A.1 Complements on Lie group

A.1.1 Orbits

We recall the definition of $G$-orbit

**Definition A.1.1 (G-orbit).** A $G$-orbit of a local transformation group is a minimal non empty group invariant subset of the manifold $M$.

A set $O \subset M$ is an orbit if it satisfies:

(i) if $x \in O$, $\epsilon \in G$ and $\exp(\epsilon v)(x)$ is defined then $\tilde{x} = \exp(\epsilon v)(x) \in O$;

(ii) if $\tilde{O} \subset O$ and $O$ satisfies part (i) then either $\tilde{O} = O$ or $\tilde{O}$ is empty.

If we are dealing with a global transformation group, for each $x \in M$ the orbit through $x$ is defined as follows

$$O_x := \{\exp(\epsilon v)(x) : \epsilon \in G\}$$

**Definition A.1.2.** Let $G$ a local group of transformations action on a manifold $M$.

1. The group $G$ acts semi-regularly if all the orbits $O$ are as the same dimension as submanifolds of $M$;

2. The group $G$ act regularly if the action is semi-regular and in addition for each $x \in M$ it exists a neighbourhood $U$ of $x$ with the property that each orbits of $G$ intersects $U$ in a pathwise connected subset.

A.1.2 The differentials

Let be $M$ and $N$ two manifolds. A smooth map $F : M \to N$ between manifolds will map smooth curves on $M$ to smooth curves on $N$ inducing thus a map between their tangent vectors. The result is a linear map $dF : TM|_x \to TN|_{F(x)}$ between the tangent spaces of the two manifolds,
called the \textit{differential of} $F$. To be more rigorous, if the parametrized curve $\phi(t)$ has tangent vector $v|_x = \phi'(t)$ at the point $x = \phi(t)$, then the image curve $\psi(t) = F(\phi(t))$ will have tangent vector $w|_y = dF(v)|_x = \psi'(t)$ at the image point $y = F(x)$. Alternatively, if we treat tangent vectors as derivations, then we can define the differential by the chain rule formula

\[
dF(v|_x)[h(y)] = v[h \circ F(x)], \quad \text{for any} \quad h : N \to \mathbb{R}
\]

In terms of local coordinates

\[
dF(v|_x) = dF \left( \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x_i} \right) = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \xi^i \frac{\partial f^j}{\partial x_i} \right) \frac{\partial}{\partial y_j} \tag{A.1}
\]

An important remark is that, in general, unless $F$ is one-to-one, its differential $dF$ does not map vector fields to vector fields. Indeed if $v$ is a vector field on $M$ and $x$ and $\tilde{x}$ are two points in $M$ with the same image $F(x) = F(\tilde{x})$ in $N$, there is no reason why $dF(v|_x)$ should necessarily agree with $dF(v|_{\tilde{x}})$. However, if $v$ is mapped to a well-defined vector field $dF(v)$ on $N$, then the two flows match up, meaning

\[
F[\exp(\epsilon v)x] = \exp(\epsilon dF(v))F(x)
\]

Moreover, the differential $dF$ respects the Lie bracket operation

\[
dF([v,w]) = [dF(v), dF(w)]
\]

whenever $dF(v)$ and $dF(w)$ are well-defined vector fields on $N$.

**Example A.1.1.** Let $M = \mathbb{R}^2$ with coordinates $(x, y)$ and let $N = \mathbb{R}$ with coordinate $s$, and let $F : \mathbb{R}^2 \to \mathbb{R}$ be any map $s = F(x, y)$. Given

\[
v|_{(x,y)} = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}
\]

then by (A.1)

\[
dF(v|_{(x,y)}) = \left\{ a \frac{\partial F}{\partial x}(x, y) + b \frac{\partial F}{\partial y}(x, y) \right\} \frac{d}{ds} |_{F(x,y)}
\]

### A.1.3 Lie Algebras

The Lie algebra can be treated more extensively considering left or right multiplication. We prefer to leave this part as complementary since it is more of an algebraic treatment of the Lie group. For a reader interested more in the application this part can be skipped without any consequence but it is of great importance for the foundational of the theory.

The most important example of Lie algebra is provided by the action of a Lie group $G$ on itself by left or right multiplication. Here, the invariant vector fields determine the Lie algebra or infinitesimal Lie group. Given $\epsilon \in G$, we let $L_\epsilon : h \mapsto \epsilon \circ h$ and $R_\epsilon : h \mapsto h \circ \epsilon$ denote the associated left and right multiplication maps. A vector field $v$ on $G$ is called left-invariant if $dL_\epsilon(v) = v$, and right-invariant if
A.1 Complements on Lie group

\[ dR_\epsilon(v) = v, \text{ for all } \epsilon \in G. \]

**Definition A.1.3.** The left (respectively right) Lie algebra of a Lie group \( G \) is the space of all left-invariant (respectively right-invariant) vector fields on \( G \).

As said to any Lie group there are associated two different algebras: \( g_L \) and \( g_R \). We have assumed we always refer to the right algebra. Anyway the reader should know a left algebra can be constructed as well.

Every right-invariant vector field \( v \) is uniquely determined by its value at the identity \( e \), because \( v|_\epsilon = dR_\epsilon(v|_e) \). Thus we can identify the right Lie algebra with the tangent space to \( G \) at the identity element, \( g \simeq TG|_e \), so \( g \) is a finite-dimensional vector space having the same dimension as \( G \).

Each Lie algebra associated with a Lie group comes equipped with a natural bracket, induced by the Lie bracket of vector fields. This follows immediately from the invariance of the Lie bracket under diffeomorphisms, which implies that if both \( v \) and \( w \) are right-invariant vector fields, so is their Lie bracket \([v, w]\).

**Definition A.1.4 (Lie-algebra).** A Lie algebra \( g \) is a vector space over some field \( \mathcal{F} \) with an additional law of combination of elements (the Lie bracket)

\[ [\cdot, \cdot] : g \times g \to g \]

such that \( \forall a, b \in \mathcal{F} \) and \( \forall v_1, v_2, v_3 \in g \) the following properties hold:

(Closure) \( [v_i, v_j] \in g \):

(Bilinearity) \( [av_i + bv_k, v_j] = a[v_i, v_j] + b[v_k, v_j] \):

(Anticommutativity) \( [v_i, v_j] = -[v_j, v_i] \):

(Jacobi identity) \( [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0 \):

for any \( v_i, v_j, v_k \in g \).

**Example A.1.2.** The Lie algebra \( \mathfrak{gl}(n) \) of the general linear group \( GL(n) \) can be identified with the space of all \( n \times n \) matrices. In terms of the coordinates provided by the matrix entries

\[ X = (x_{ij}) \in GL(n) \]

the left-invariant vector field associated with a matrix

\[ A = (a_{ij}) \in \mathfrak{gl}(n) \]
has the explicit formula
\[ \hat{v}_A = \sum_{i,j,k=1}^{n} x_{ik} a_{kj} \frac{\partial}{\partial x_{ij}} \]  
(A.2)

The Lie bracket of two such vector fields is
\[ [\hat{v}_A, \hat{v}_B] = \hat{v}_C \]

where \( C = AB - BA \), so the left-invariant Lie bracket on \( GL(n) \) can be identified with the standard matrix commutator
\[ [A, B] = AB - BA. \]

On the other hand, the right-invariant vector field associated with a matrix \( A \in gl(n) \) is given by
\[ v_a = \sum_{i,j,k=1}^{n} x_{ik} a_{kj} \frac{\partial}{\partial x_{ij}} \]  
(A.3)

Now the Lie bracket is
\[ [\hat{v}_A, v_B] = v_C \]

where \( \hat{C} = -C = BA - AB \) is the negative of the matrix commutator. Thus, the matrix formula for the Lie algebra bracket on \( gl(n) \) depends on whether we are dealing with its left-invariant or right-invariant version.

### A.1.4 The exponential map

From a philosophical perspective, the exponential function may be viewed as a link between the seemingly contradictory positions of Heraclitus on the one side and Parmenides on the other. While the time-dependent function \( t \to T(t) \) reflects the aspect of permanent change in a deterministic autonomous system, its generator \( A \) stands for the eternal, timeless principle behind the system. The exponential function ties both aspects together.\(^1\)

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Given a right-invariant vector field \( v \in g \) on the Lie group \( G \), we let
\[ \exp(\epsilon v) : G \to G \]
denote the associated flow. Applying the flow to the identity element \( e \) serves to define the one-parameter subgroup
\[ \exp(\epsilon v) \equiv \exp(\epsilon v)e \]
the vector field \( v \) is known as the infinitesimal generator of the subgroup. The notation is not ambiguous, since the flow through any \( g \in G \) is the same as left multiplication by the elements of the

\(^1\)The author may wish to thank S.B. for the hint.
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subgroup, so \( \exp(\epsilon v) \) can be interpreted either as a flow or as a group multiplication. Vice versa, the infinitesimal generators of the action of \( G \) on itself by right multiplication is the Lie algebra \( g_L \) left-invariant vector fields. This interchange of the role of infinitesimal generators and invariant vector fields is one of the interesting peculiarities of Lie group theory. Finally, note that although the left- and right-invariant vector fields associated with a given tangent vector \( v \in TG|_{e} \) are (usually) different, and have different flows, nevertheless the associated one-parameter groups coincide

\[
\exp(\epsilon v_L) = \exp(\epsilon v_R)
\]

Example A.1.3. Let us consider again the general linear group \( GL(n) \) previously discussed.

The flow corresponding to the right-invariant vector field \( v_A \) (A.3) is given by left multiplication by the usual matrix exponential

\[
\exp(\epsilon v_A)X = e^{\epsilon A}X
\]

Conversely, the flow corresponding to the left-invariant vector field \( \hat{v}_A \) given by (A.2) is given by right multiplication

\[
\exp(\epsilon \hat{v}_A)X = X e^{\epsilon A}
\]

In either case, the one-parameter subgroup generated by the vector field associated with a matrix \( A \in \text{gl}(n) \) is the matrix exponential

\[
\exp(\epsilon \hat{v}_A) = \exp(\epsilon v_A) = e^{\epsilon A}
\]

\( \exp(tvA) = \exp(tbvA) = e^{tA} \).

From now on we shall restrict our attention to the right-invariant vector fields, so that \( g \) will always denote the right Lie algebra of the Lie group \( G \). Evaluation of the flow \( \exp(\epsilon v) \) at \( \epsilon = 1 \) for each \( v \in g \) serves to define the exponential map \( \exp : g \to G \). Since \( \exp(0) = e \), \( d\exp(0) = 0 \), the exponential map defines a local diffeomorphism in a neighbourhood of \( 0 \in g \). Consequently, all Lie groups having the same Lie algebra look locally the same in a neighbourhood of the identity; only their global topological properties are different.

A.1.5 Infinitesimal groups actions

Just as a one-parameter group of transformations is generated as the flow of a vector field, so a general Lie group of transformations \( G \) acting on a manifold \( M \) will be generated by a set of vector fields on \( M \), known as the infinitesimal generators of the group action.

Each infinitesimal generators flow coincides with the action of the corresponding one-parameter subgroup of \( G \). Specifically, let us suppose \( G \) is a local Lie group of transformations acting on a manifold \( M \) via \( \exp(\epsilon v)(x) \). Then if \( v \in g \) we define \( \psi(v) \) to be the vector field on \( M \) whose flow coincides with the action of the one-parameter subgroup \( \exp(\epsilon v) \) of \( G \) on \( M \).

Thus

\[
\psi(v)|_x = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Psi_{\exp(\epsilon v)}(x) = d\Psi_{v|_x}(x)
\]  

(A.4)
Note now that
\[ \Psi(x) \circ R_\epsilon(\delta) = \Psi_{\epsilon \delta}(x) = \Psi_\delta(\Psi_\epsilon(x)) \]
Thus we have
\[ d\Psi|_x = d\Psi_{\epsilon}(\Psi_\epsilon(x)) = \psi|_{\Psi_\epsilon(x)} \]
It follows immediately from the properties of the Lie bracket that
\[ [\psi(v_i), \psi(v_j)] = \psi([v_i, v_j]) \]

**Theorem A.1.5.** *Let us consider \( \{ w_j \}_{j=1}^r \) vector field on a manifold \( M \) satisfying
\[
[w_i, w_j] = \sum_{k=1}^r C_{ij}^k w_k, \quad i, j = 1, \ldots, r
\]
for certain constants \( C_{ij}^k \). Then there is a Lie group \( G \) whose Lie algebra has the given \( C_{ij}^k \) as structure constants relative to some basis \( \{ v_j \}_{j=1}^r \) and a local group action of \( G \) on \( M \) such that \( \psi(v_i) = w_i \) where \( \psi \) is defined as in (A.4).*

The differential \( d\Psi \) preserves the Lie bracket between vector fields; therefore the resulting vector fields form a finite-dimensional Lie algebra of vector fields on the manifold \( M \), satisfying the same commutation relations as the right Lie algebra \( g \) of \( G \).

Just as every Lie algebra generates a corresponding Lie group, given a finite-dimensional Lie algebra of vector fields on a manifold \( M \), we can always reconstruct a (local) action of the corresponding Lie group via the exponentiation process.

**Theorem A.1.6.** *Let \( g \) be a finite-dimensional Lie algebra of vector fields on a manifold \( M \). Let \( G \) denote a Lie group having Lie algebra \( g \). Then there is a local action of \( G \) whose infinitesimal generators coincide with the given Lie algebra.*
Appendix B

Appendix

B.1 Fundamental solutions for parabolic equations

Ah, Avner Avner...il buon vecchio Avner.
Ah, Avner Avner...good old Avner.

Anonymous

We are here to give some general notions on linear second-order parabolic equations and fundamental solutions. For a deeper treatment see Evans [Eva10] and Friedman [Fri64].

Second-order parabolic equations are natural generalizations of the heat equation seen in section ??.

B.1.1 Definitions

We will assume further \( U \) to be an open, bounded subset of \( \mathbb{R}^N \) and we set \( U_T = U \times [0, T] \) for some fixed \( T > 0 \).

We will then study the Cauchy-Dirichlet problem (CD problem)

\[
\begin{cases}
    u_t + Lu = f, & \text{in } U_T \\
    u = 0, & \text{on } \partial U \times [0, T] \\
    u = g, & \text{on } U \times \{t = 0\}
\end{cases}
\] (B.1)

where \( f : U_T \to \mathbb{R} \) and \( g : U \to \mathbb{R} \) are given functions and \( u : \bar{U}_T \to \mathbb{R} \) is the unknown, \( u = u(x, t) \) with \( x \in \mathbb{R}^N \) and \( t \in [0, T] \). For each time \( t \) \( L \) denotes a second-order linear differential operator either in the diverge form

\[
Lu = -\sum_{i,j=1}^{N} (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{N} b^i(x,t)u_{x_i} + c(x,t)u
\] (B.2)
or the non-divergence form

\[ Lu = - \sum_{i,j=1}^{N} a^{ij}(x,t)x_i x_j + \sum_{i=1}^{N} b^{i}(x,t)x_i + c(x,t)u \]  \hspace{1cm} (B.3) 

both for given coefficients \( a^{ij}, b^{i} \) and \( c \).

**Definition B.1.1.** We say that the partial differential operator \( \partial_t + L \) is (uniformly) parabolic if there exists a constant \( K > 0 \) s.t.

\[ \sum_{i,j=1}^{N} a^{ij}(x,t)\xi_i \xi_j \geq K|\xi|^2 \]

\( \forall (x,t) \in U_T, \xi \in \mathbb{R}^N. \)

**Remark B.1.2.** Let us notice that for each fixed time \( 0 \leq t \leq T \) the operator is uniformly elliptic in the spacial variable \( x \).

**Remark B.1.3.** If we take \( a^{ij} \equiv \delta^{ij}, b^{i} \equiv c \equiv f \equiv 0 \) such that \( L = -\Delta \), where we have denoted by \( \delta^{ij} \) the Kronecker delta and \( \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \) the Laplacian operator, the PDE becomes the well-known heat equation.

In general a second order parabolic operator describes the time-evolution of the density of some quantity \( u \). In our application \( u \) will mostly represents the payoff function of some particular financial instrument.

### B.1.2 The fundamental solution of the heat equation

We will now focus on the \( N \)-dimensional heat equation

\[
\begin{cases}
  u_t - \Delta u = 0, & \text{in } \mathbb{R}^N \times [0, \infty] \\
  u = g, & \text{in } \mathbb{R}^N \times \{ t = 0 \}
\end{cases}
\]  \hspace{1cm} (B.4)

In this section it will come to light why we deal with fundamental solutions and in particular they play such an important role in studying parabolic equations.

To help intuition, think for instance of this special solution as the concentration of a substance of total mass \( q \) and suppose we want to keep the total mass equal to \( q \) at any time.

We observe that the heat equation involves one derivative with respect to the time variable \( t \) but two derivatives with respect to the space variable \( x \). Consequently if \( u \) solves (B.4) then so does \( u(\lambda x, \lambda^2 t) \) for \( \lambda \in \mathbb{R} \). This scaling indicates the ratio \( \frac{|x|^2}{t} \) plays an important role for the heat equation (the astute reader will notice that we have already meet this symmetry in section ??).

We will therefore look for a solution of (B.4) having the form \( u(x,t) = v\left( \frac{|x|^2}{t} \right) \). Although this approach eventually leads to what we want, it is quicker to seek a solution \( u \) having the special form

\[ u(x,t) = \frac{1}{t^\alpha} v\left( \frac{x}{t^\beta} \right) \]  \hspace{1cm} (B.5)
where $\alpha$, $\beta$ and $v$ are to be found. We are here asking

$$u(x, t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

Inserting now (B.5) into (B.4) we obtain

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \quad (B.6)$$

for $y := t^{-\beta} x$. In order now to transform (B.6) into an expression involving the variable $y$ alone we take $\beta = \frac{1}{2}$. Therefore we have

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0 \quad (B.7)$$

If we guess further that $v$ is radial, i.e. $v(y) = w(|y|)$ (B.7) becomes

$$\alpha w + \frac{1}{2} r w' + w' + w'' + \frac{n-1}{r} w' = 0$$

for $r = |y|$. If we now set $\alpha = \frac{n}{2}$ we obtain

$$\left(r^{n-1} w\right)' \frac{1}{2} \left(r^n w\right)' = 0$$

Thus

$$r^{n-1} w' + \frac{1}{2} r^n w = a$$

Assuming now $\lim_{r \to \infty} w, w' = 0$ we can conclude $a = 0$. Whence

$$w' = - \frac{1}{2} r w$$

Then for some constant $b$

$$w = be^{-\frac{r^2}{4}} \quad (B.8)$$

Combining now (B.5) and (B.8) and our choice of $\alpha$ and $\beta$ we conclude that

$$\frac{b}{t^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}} \quad (B.9)$$

solves the heat equation (B.4).

**Definition B.1.4.** The function

$$\Phi(x, t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}}, & (x \in \mathbb{R}^N, \ t > 0) \\ 0, & (x \in \mathbb{R}^N, \ t < 0) \end{cases} \quad (B.10)$$

is called the fundamental solution for the heat equation.

The choice of the normalizing constant $(4\pi)^{-n/2}$ is dictated by the following
Lemma B.1.5 (Integral of fundamental solution). For each time $t > 0$

$$
\int_{\mathbb{R}^N} \Phi(x, t) dx = 1
$$

We want now to use $\Phi$ to recover a solution to the C-D problem (B.4). Let us notice that the function $(x, t) \mapsto \Phi(x, t)$ solves the heat equation away from the singularity at $(0, 0)$ and so does $(x, t) \mapsto \Phi(x - y, t)$ for each fixed $y \in \mathbb{R}^N$. Consequently the convolution

$$
u(x, t) = \int_{\mathbb{R}^N} \Phi(x - y, t)g(y)dy \quad \text{(B.11)}$$

should also be a solution.

Theorem B.1.6 (Solution of Cauchy-problem). Assume $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and define $u$ by (B.11). Then

(i) $u \in C^\infty(\mathbb{R}^N \times ]0, \infty[)$;

(ii) $u_t(x, t) - \Delta u(x, t) = 0$;

(iii) $\lim_{(x, t) \to (x^0, 0)} u(x, t) = g(x^0)$ for each $x^0 \in \mathbb{R}^N$.

Remark B.1.7. In view of Theorem B.1.6 we will write

$$
\begin{cases}
\Phi_t - \Delta \Phi = 0, & \text{in } \mathbb{R}^N \times ]0, \infty[ \\
\Phi = \delta_0, & \text{on } \mathbb{R}^N \times \{t = 0\}
\end{cases}
$$

(B.13)

$\delta_0$ denoting the Dirac mass on $\mathbb{R}^N$ giving unit mass to the point 0.

B.1.3 Fundamental solutions

Definition B.1.8. We say that $u = u(x, t)$ is a solution of $\partial_t u + Lu = 0$ in some region $\Omega$ if all derivatives of $u$ which occur in $\partial_t u + Lu = 0$ are continuous functions in $\Omega$ and $\partial_t u(x, t) + Lu(x, t) = 0$ at each point $(x, t) \in \Omega$.

On the other side we define a fundamental solution as

Definition B.1.9. A fundamental solution of $\partial_t u + Lu = 0$ is a function $\Gamma(x, t; \xi, \tau)$ defined for all $(x, t) \in U_T$, $(\xi, \tau) \in U_T$, $t > \tau$ which satisfies the following conditions:

(i) for fixed $(\xi, \tau) \in U_T$ it satisfies, as a function of $(x, t)$ ($x \in U$, $\tau < t \leq T$) the equation $\partial_t u + Lu = 0$;
(ii) for every continuous function \( f(x) \in \bar{U} \), if \( x \in U \) then

\[
\lim_{t \searrow \tau} \int_{\bar{U}} \Gamma(x, t; \xi, \tau) f(\xi) d\xi = f(x)
\]

We can think of the fundamental solution as a unit source solution: \( \Gamma(x, t; \xi, \tau) \) for fixed \((\xi, \tau)\) gives the concentration at the point \(x\) at time \(t\), generated by the diffusion of a unit mass initially concentrated at the origin (let us think at remark B.1.7 with initial condition \( \delta_0 \)).

From another point of view, if we imagine a unit mass composed of a large number \(N\) of particles, \( \Gamma(x, t; \xi, \tau) \) gives the probability that a single particle is placed between \(x\) and \(\xi\) at time \(t\) or equivalently the percentage of particles inside the interval \((x, \xi)\) at time \(t\). This suggests us that the fundamental solution is the perfect connection between PDE and the transition density function of a process associated to the PDE.
Appendix C

Appendix

C.1 The Longstaff Model

The Longstaff model has been proposed by Longstaff as an alternative model to the CIR. It has been shown empirically that this model can fit better data for short rate.

Given the PDE

$$u_t + \frac{1}{2}\rho^2x^2\gamma u_{xx} + (\alpha + \beta x - \lambda \rho x^\gamma)u_u - xu = 0 \quad (C.1)$$

The solution $u(x, t)$ of this PDE together with the final condition $u(x, T) = 1$ gives the price for a Zero-Coupon bond with maturity $T$. The parameter $\lambda$ is called the market price of risk. It represents an extra increase in the expected rate of return on a bond per additional unit of risk. In general it may depends upon both $x$ and $t$.

We will in this section concentrate on the case $\gamma = \frac{1}{2}$ and $\alpha = \frac{\rho^2}{4}$ which gives us the Longstaff equation.

It is not convenient to proceed with the method developed by Craddock since this equation solves the first Riccati equation only for certain parameters. This narrow any possible use of this model. Anyway another approach is possible. In fact we will show now that this equation can be mapped in the standard heat equation.

We will now look for any possible symmetries of the PDE

$$u_t + \frac{1}{2}\rho^2 x u_{xx} + (\frac{\rho^2}{4} + \beta - \lambda \rho \sqrt{x})u_u - xu = 0 \quad (C.2)$$

Applying the standard Lie algorithm we can see that the symmetric group for the equation (C.2) is
spanned by
\[
\begin{align*}
\mathbf{v}_1 &= \partial_t t \\
\mathbf{v}_2 &= u \partial_u \\
\mathbf{v}_3 &= e^{\alpha t} \left( \sqrt{x} \partial_x - \frac{\beta - \mu}{\rho^2} \left( \frac{\rho \mu}{\sqrt{x}} + \sqrt{x} \right) u \partial_u \right) \\
\mathbf{v}_4 &= e^{-\alpha t} \left( \sqrt{x} \partial_x + \frac{\beta + \mu}{\rho^2} \left( \frac{\rho \mu}{\sqrt{x}} - \sqrt{x} \right) u \partial_u \right) \\
\mathbf{v}_5 &= e^{\alpha t} \left( x - \frac{\beta \lambda \rho}{\mu^2} \sqrt{x} \right) \partial_x + \frac{\beta}{\rho} \partial_t + (a_0(\mu) + a_1(\mu) \sqrt{x} + xa_2(\mu)) u \partial_u \\
\mathbf{v}_6 &= e^{\alpha t} \left( x - \frac{\beta \lambda \rho}{\mu^2} \sqrt{x} \right) \partial_x - \frac{1}{\rho} \partial_t + (a_0(-\mu) + a_1(-\mu) \sqrt{x} + xa_2(-\mu)) u \partial_u
\end{align*}
\] (C.3)

with
\[
\mu = \sqrt{\beta^2 + 2 \rho^2}, \quad a_0(\zeta) = \frac{\beta - \zeta}{4 \zeta} - \frac{\lambda^2 (\rho^2 + (\beta - \zeta) \zeta)}{\zeta^3}, \quad a_1(\zeta) = \frac{\lambda (\beta - \zeta)^2}{\rho^2 \zeta^2}, \quad a_2(\zeta) = \frac{\zeta - \beta}{\rho^2}
\]

We have already anticipated equation (C.2) can be mapped into the standard heat equation by an invertible transformation of the form
\[
z = f(x, t), \quad \tau = g(t), \quad w = h(x, t) u
\] (C.4)

We have therefore to solve the Cauchy problem
\[
\begin{align*}
w_{\tau} &= w_{zz} \\
w(z, 0) &= \eta(\zeta)
\end{align*}
\] (C.5)

It is well-known that a solution is given by
\[
w(z, \tau) = \frac{1}{2 \sqrt{\pi \tau}} \int_{-\infty}^{\infty} \eta(\zeta) \exp \left( - \frac{(z - \zeta)^2}{4 \tau} \right) d\zeta
\] (C.6)

Obviously the solution obtained in (C.6) is transformable back into a solution of (C.2) by the inverse of (C.4).

In order to construct (C.4) we will use \(\mathbf{v}_3\) and \(\mathbf{v}_5\). The solution so found of (C.2) will be thus the Zero-Coupon bond price under the Longstaff model.

In particular we have to use
\[
f(x, t) = K_1 + e^{-\alpha t} \left( 2 \sqrt{x} - \frac{2 \beta \lambda \rho}{\mu^2} \right)
\]
\[
g(t) = \rho^2 \left( e^{-\alpha t} - e^{-\alpha T} \right)
\]
\[
h(x, t) = K_2 \exp \left( \Lambda t + \frac{(\beta - \mu) x}{\rho^2} + \frac{2 \lambda (\beta - \mu) \sqrt{x}}{\rho \mu} \right)
\]
\[ \eta(\zeta) = \exp \left( \Lambda T + (\mu - \beta) \left( \frac{e^{\mu T/2}\lambda(\mu + \beta)(K_1 - \zeta)}{\rho \mu^2} - \frac{e^{\mu T}(K_1 - \zeta)^2}{4\rho^2} - \frac{\beta \lambda^2(\beta + 2\mu)}{\mu^4} \right) \right) \]

\[ u(x, t) = \frac{e^{\varphi(x, t)}}{\sqrt{1 + \left( \frac{\beta - \mu}{2\lambda \rho \psi_2(T - t)} \right)^2}} \]

where \( K_1 \) and \( K_2 \) are some constants, \( \Lambda = \frac{\lambda^2(\beta^2 + \rho^2)}{\mu^2} + \frac{\mu - \beta}{4\lambda \rho} - \lambda^2 \) and

\[ \varphi(x, t) = \frac{1}{1 + \left( \frac{\beta - \mu}{2\lambda \rho \psi_2(T - t)} \right)^2} \left( 2\lambda \rho \psi_1(T - t)^2 \sqrt{x} + x \psi_2(T - t) + \right. \]

\[ \left. + \frac{\Lambda}{2} (\beta \psi_2(T - t) + \mu \psi_3(T - t)) (T - t) - \frac{\lambda^2 \rho^2}{\mu^2} (2\beta \psi_1(T - t)^2 + \psi_2(T - t)) \right) \]

and

\[ \psi_1(y) = \frac{1 - e^{\mu y}}{\mu}, \quad \psi_2(y) = \frac{1 - e^{\mu y}}{\mu}, \quad \psi_3(y) = \frac{1 + e^{\mu y}}{\mu} \]
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